ALTERNATIVE PROOF ON THE CROSSING NUMBER OF $K_{1,1,3,N}$

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ABSTRACT
The main aim of the paper is to give the crossing number of join product $G+D_n$ for the connected graph G of order five isomorphic with the complete tripartite graph $K_{1,1,3}$, where D_n consists on n isolated vertices. The proof of the crossing number of $K_{1,1,3,n}$ was published by very rather unclear discussion of cases by Ho in [5]. In our proofs, it will be extend the idea of the minimum numbers of crossings between two different subgraphs from the set of subgraphs which do not cross the edges of the graph G onto the set of subgraphs which cross the edges of the graph G exactly once. The methods used in the paper are new, and they are based on combinatorial properties of cyclic permutations. Finally, by adding one edge to the graph G, we are able to obtain the crossing number of the join product with the discrete graph D_n for one new graph.

Keywords: *graph, drawing, crossing number, join product, cyclic permutation*

1. INTRODUCTION

Over the last years, some results concerning crossing numbers of join products of two graphs have been obtained. So, the purpose of this article is to extend the known results concerning this topic. It is well known that the problem of reducing the number of crossings in the graph was studied in a lot of areas, and the most prominent area is VLSI technology. The lower bound on the chip area is determined by crossing number and by number of vertices of the graph. The investigation on the crossing number of graphs is a classical and very difficult problem provided that an computing of the crossing number of a given graph in general is NP-complete problem.

In the paper, we will use definitions and notations of the crossing numbers of graphs like in [8]. Some proofs are based on the Kleitman's result on crossing numbers of the complete bipartite graphs [6]. More precisely, he proved that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6.$$

The exact values for the crossing numbers of $G+D_n$ for all graphs G of order at most four are given in [11]. Also, the crossing numbers of the graphs $G+D_n$ are known for few graphs G of order five and six, see [2], [7], [10], [11], [12], [13], and [14]. In all these cases, the graph G is connected and contains at least one cycle.

The methods used in the paper are new, and they are based on combinatorial properties of the cyclic permutations. The similar methods were partially used first time in the papers [4], and [12]. In [2], [3], and [13], the properties of cyclic permutations are also verified by the help of software in [1]. According to our opinion the methods used in [7], [10], and [11], do not allow to establish the crossing number of the join product $G+D_n$. Let G be the connected graph of order five isomorphic with the complete tripartite graph $K_{1,1,3}$. We consider the join product of G with the discrete graph on n vertices denoted by D_n. The graph $G+D_n$ consists of one copy of the graph G and of n vertices $t_1, t_2, \ldots, t_n$, where any vertex t_i, $i = 1, 2, \ldots, n$, is adjacent to every vertex of G. Let T^i, $1 \leq i \leq n$, denote the subgraph induced by the five edges incident with the vertex t_i. Thus, the graph $T^1 \cup \cdots \cup T^n$ is isomorphic with the complete bipartite graph $K_{5,n}$ and

$$G+D_n = G \cup K_{5,n} = G \cup \left(\bigcup_{i=1}^{n} T^i \right).$$

2. CYCLIC PERMUTATIONS

In the paper, we will use the same definitions and notation for cyclic permutations and the corresponding configurations for a good drawing D of the graph $G+D_n$ like in [3], and [13]. The *rotation* $\mathrm{rot}_D(t_i)$ of a vertex t_i in the drawing D like the cyclic permutation that records the (cyclic) counter-clockwise order in which the edges leave t_i have been defined in [4]. Let us denote by R_D the set of subgraphs T^i, $i \in \{1, \ldots, n\}$, for which $\mathrm{cr}_D(G, T^i) = 0$. Since the set R_D can be empty, in the good drawing D of the graph $G+D_n$, we will have to discus also some restricted drawings for other subgraphs. Thus, let S_D be the set of subgraphs T^i, $i \in \{1, \ldots, n\}$, for which $\mathrm{cr}_D(G, T^i) = 1$. Moreover, let F^i denote the subgraph $G \cup T^i$ for $T^i \in R_D \cup S_D$. Due to arguments in the proof of the main Theorem 3.1, if we want to obtain a drawing of $G+D_n$ with the smallest crossing number, then the set $R_D \cup S_D$ must be nonempty. Hence, we will deal with only drawings of the graph G with a possibility of an existence of a subgraph $T^i \in R_D \cup S_D$.

Since the graph G consists from the edge disjoint cycles C_3 and C_4, we only need to consider possibilities of crossings between subdrawings of subgraphs C_3 and C_4. Of course, the edges of the cycle C_4 can cross itself in the considered subdrawings. Let us assume first a good subdrawing of G in which the edges of C_4 do not cross each other. In this case, we obtain one planar drawing shown in Fig. 1(a). If we consider a good subdrawing of G with one crossing among edges of the cycle C_4, then the edges of C_3 do not cross the edges of C_4 in two cases which are shown in Fig. 1(b), and (c). (The vertex notation of the graph G will be justified later.) If the edges of C_4 are crossed at least once by the edges of C_3, then there are only three possibilities according to the considered good subdrawing of G and they are showed in Fig. 1(d), (e), and (f).

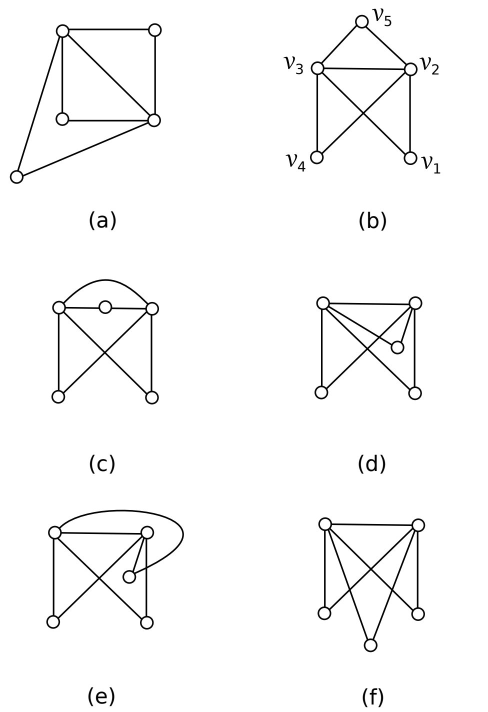

Fig. 1 One planar drawing of G and five drawings of G with $\mathrm{cr}_D(G) \geq 1$

Moreover, due to Lemma (3.1), we obtain at least the considered crossing number of the graph $G+D_n$ for the cases of drawing of the graph G in such a way as shown in Fig. 1(a), (c), (d), and (e). Assume a good drawing D of the graph $G+D_n$ in which the edges of G cross each other exactly once. In this case, without loss of generality, we can choose the vertex notation of the graph in such a way as shown in Fig. 1(b). If there is a $T^i \in R_D$, then the subgraph F^i is represented by $\mathrm{rot}_D(t_i) = (14352)$. Our aim is to list all possible rotations $\mathrm{rot}_D(t_j)$ which can appear in D if the edges of T^j cross the edges of G exactly once. Since there are only two possibilities for both edges t_jv_2 and t_jv_3 how to cross the edges of graph G exactly once, we have four different possible configurations of the subgraph F^j denoted as A_1, A_2, A_3, and A_4, i.e. $\mathrm{rot}_D(t_j) = A_k$ for $k = 1,2,3,4$.

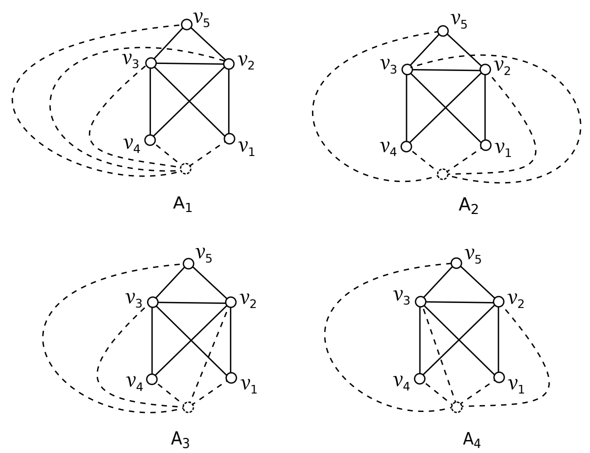

Fig. 2 Drawings of four possible configurations from $\mathscr{M}$ of the subgraph F^i

As for our considerations it does not play role which of the regions is unbounded, assume the drawings shown in Fig. 2. In the rest of the paper, each cyclic permutation will be represented by the permutation with 1 in the first position. Thus, the configurations A_1, A_2, A_3, and A_4 are represented by the cyclic permutations (14325), (14532), (12435), and (13452), respectively. Of course, in a fixed drawing of the graph $G+D_n$, some configurations from $\mathscr{M}$ do not must appear. We denote by $\mathscr{M}_D$ the set of all configurations that exist in the drawing D belonging to the set $\mathscr{M} = \{A_1, A_2, A_3, A_4\}$.

Let us note that we are able to extend the idea of the minimum numbers of crossings between two subgraphs T^i and T^j from the set R_D onto the set S_D. Let X, Y be two configurations from $\mathscr{M}_D$. We shortly denote by $\mathrm{cr}_D(X,Y)$ the number of crossings in D between T^i and T^j for different $T^i, T^j \in S_D$ such that F^i, F^j have configurations X, Y, respectively. Finally, let $\mathrm{cr}(X,Y) = \min\{\mathrm{cr}_D(X,Y)\}$ over all good drawings of the graph $G+D_n$ with $X, Y \in \mathscr{M}_D$. Our aim is to establish $\mathrm{cr}(X,Y)$ for all pairs $X, Y \in \mathscr{M}$.

The configurations A_1 and A_2 are represented by the cyclic permutations (14325) and (14532), respectively. Since the minimum number of interchanges of adjacent elements of (14325) required to produce cyclic permutation $\overline{(14532)} = (12354)$ is two, any subgraph T^j with the configuration A_2 of F^j crosses the edges of T^i at least twice, i.e. $\mathrm{cr}(A_1,A_2) \geq 2$. The same reason gives $\mathrm{cr}(A_1,A_3) \geq 2$, $\mathrm{cr}(A_1,A_4) \geq 2$, $\mathrm{cr}(A_2,A_3) \geq 2$, $\mathrm{cr}(A_2,A_4) \geq 2$, and $\mathrm{cr}(A_3,A_4) \geq 2$. Moreover, by a discussion of possible subdrawings, we can verify that $\mathrm{cr}(A_3,A_4) \geq 4$. Let F^i and F^j be two different subgraphs having the configurations A_3 and A_4, respectively. Let us start with the subdrawing $D(T^i \cup T^j \setminus v_5)$ of $T^i \cup T^j$ induced by the edges incident with the vertices v_1, v_2, v_3, and v_4 shown in Fig. 2. Hence, if we suppose that $\mathrm{cr}_D(T^i,T^j) = 2$, then there is no crossing in the subdrawing $D(T^i \cup T^j \setminus v_5)$ provided by the properties of the cyclic permutations. This forces a contradiction, since the edges t_iv_2 and t_jv_3 have to cross the edges v_1v_3 and v_2v_4 of the graph G in the subdrawing $D(F^i \cup F^j \setminus v_5)$, respectively. Consequently, the Woodall's result $\mathrm{cr}_D(T^i,T^j) = Q(\mathrm{rot}_D(t_i), \mathrm{rot}_D(t_j)) + 2k$ for some nonnegative integer k in [15] forces $\mathrm{cr}_D(T^i,T^j) \geq 4$, where Q have been defined in [13]. Clearly, also $\mathrm{cr}(A_k,A_k) \geq 4$ for any $k = 1,2,3,4$. Thus, all lower-bounds of number of crossing of configurations from $\mathscr{M}$ are summarized in symmetric Table 1. (Here, A_k and A_l are configurations of the subgraphs F^i and F^j, where $k,l \in \{1,\dots,4\}$.)

Table 1 The necessary number of crossings between T^i and T^j

−	A_1	A_2	A_3	A_4
A_1	4	2	2	2
A_2	2	4	2	2
A_3	2	2	4	4
A_4	2	2	4	4

3. THE CROSSING NUMBER OF $G+D_N$

Two vertices t_i and t_j of $G+D_n$ are *antipodal* in a drawing of $G+D_n$ if the subgraphs T^i and T^j do not cross. A drawing is *antipodal-free* if it has no antipodal vertices. In the rest of the paper, each considered drawing of the graph $G+D_n$ will be assumed antipodal-free. In the proof of Theorem 3.1, the following assertion related to some restricted drawings of the graph $G+D_n$ is needed.

Lemma 3.1. *Let D be a good drawing of $G+D_n$, $n>2$, and let $D(G)$ be the subdrawing of the graph G induced by D. If there is a subgraph H of the graph G such that, in the subdrawing $D(G)$, H is isomorphic with the cycle C_3 which separates the other vertices of the graph G, then there are at least $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor$ crossings in D.*

Proof. By assumption, let us consider that H be a subgraph of the graph G in the subdrawing $D(G)$ for which H is isomorphic with the cycle $C_3=c_1c_2c_3c_1$ divided the plane into two regions in such a way that all $5-3=2$ vertices of the graph G not belonging to C_3 lie in different regions. This fact implies that, in the good drawing D of the graph $G+D_n$, there is no $T^i\in R_D$. Hence, any subgraph T^i crosses the edges of C_3 at least once for any $i=1,\dots,n$. This forces $\mathrm{cr}_D(G_H+D_n)\geq n$, where G_H is the subgraph of G with the vertex set $V(G)$, and the edge set $\{c_1c_2,c_1c_3,c_2c_3\}$. Let us denote by G_{G-H} the subgraph of G with the vertex set $V(G)$, and the edge set $E(G)\setminus\{c_1c_2,c_1c_3,c_2c_3\}$. Since the exact value for the crossing number of the graph $G_{G-H}+D_n$ is given in [12], i.e. $\mathrm{cr}(G_{G-H}+D_n)=4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor$, then $\mathrm{cr}_D(G_{G-H}+D_n)\geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+\lfloor\frac{n}{2}\rfloor$. Consequently, we have $\mathrm{cr}_D(G+D_n)=\mathrm{cr}_D(G_{G-H}+D_n)+\mathrm{cr}_D(G_H+D_n)\geq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor$.

Now we are able to prove the main results of the paper. We will compute the exact values of crossing numbers of the small graphs in this paper using algorithm located on the website `http://crossings.uos.de/`. It uses an ILP formulation, based on Kuratowski subgraphs, and solves it via branch-and-cut-and-price. The system also generates verifiable formal proofs. So, we obtain the following result.

Lemma 3.2. $\mathrm{cr}(G+D_2)=3$.

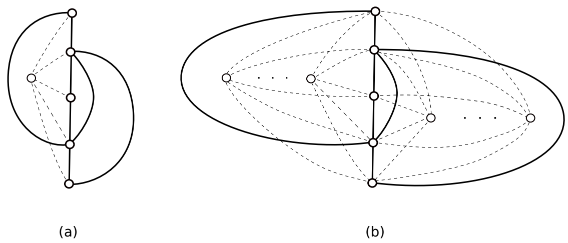

Fig. 3 The good drawings of $G+D_1$ and of $G+D_n$

Theorem 3.1. $\mathrm{cr}(G+D_n)=4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor$ *for any* $n\geq 1$.

Proof. In Fig. 3(b) there is the drawing of the graph $G+D_n$ with $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor$ crossings. Thus, $\mathrm{cr}(G+D_n)\leq 4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor$. We prove the reverse inequality by induction on n. The graph $G+D_1$ contains a subdivision of $K_{3,3}$, and therefore $\mathrm{cr}(G+D_1)\geq 1$. So, $\mathrm{cr}(G+D_1)=1$ by the good drawing of $G+D_1$ in Fig. 3(a). By Lemma 3.2 the result is true for $n=2$. Suppose now that for $n\geq 3$, there is a drawing D of $G+D_n$ with less than $4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+n+\lfloor\frac{n}{2}\rfloor$ crossings, and let

$$\mathrm{cr}(G+D_m)\geq 4\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor+m+\left\lfloor\frac{m}{2}\right\rfloor \text{ for any } m<n.$$

Our assumption on D together with the known result $\mathrm{cr}(K_{5,n})=4\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ implies that

$$\mathrm{cr}_D(G)+\mathrm{cr}_D(G,K_{5,n})<n+\left\lfloor\frac{n}{2}\right\rfloor.$$

Hence, if we will assume that the set R_D is empty, then for $s=|S_D|$ we obtain

$$\mathrm{cr}_D(G)+1s+2(n-s)<n+\left\lfloor\frac{n}{2}\right\rfloor. \qquad (1)$$

This forces $s\geq 2$, and $s\geq n-\lfloor\frac{n}{2}\rfloor+1+\mathrm{cr}_D(G)$. According to Lemma 3.1 we will discuss only the following cases:

Case 1: $\mathrm{cr}_D(G)=1$. Let us consider a subdrawing of the graph G as in Fig. 1(b). The reader can easy to verify over all possible drawings D with the nonempty set R_D that if $T^i\in R_D$, then the subgraph F^i is represented by $\mathrm{rot}_D(t_i)=(14352)$ and $\mathrm{cr}_D(G\cup T^i,T^j)\geq 4$ for any subgraph T^j, $j\neq i$. Hence, by fixing the graph $G\cup T^i$,

$$\mathrm{cr}_D(G+D_n)=\mathrm{cr}_D(K_{5,n-1})+\mathrm{cr}_D(K_{5,n-1},G\cup T^i)+$$
$$+\mathrm{cr}_D(G\cup T^i)\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+4(n-1)+1\geq$$
$$\geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor.$$

In addition, let us suppose that the set R_D is empty. Thus, we will deal with the configurations belonging to the nonempty set $\mathscr{M}_D$ due to condition (1).

(a) $A_j\in\mathscr{M}_D$ for some $j\in\{3,4\}$. Let us first show that the considered drawing D must be antipodal-free. Of course, if T^k and T^l are two different subgraphs from the nonempty set S_D, then the vertices v_k and v_l can not be antipodal according to the positive values in Table 1. As a contradiction we can suppose that $\mathrm{cr}_D(T^k,T^l)=0$, and at least one of the subgraphs T^k,T^l is not included in the set S_D. Since we assume that the set R_D is empty, then $\mathrm{cr}_D(G,T^k\cup T^l)\geq 3$. Moreover, the known fact that $\mathrm{cr}(K_{5,3})=4$ implies that any T^m, $m\neq k,l$, crosses $T^k\cup T^l$ at least four times. So, for the number of crossings, in D, we have

$$\mathrm{cr}_D(G+D_n)=\mathrm{cr}_D(G+D_{n-2})+\mathrm{cr}_D(T^k\cup T^l)+$$
$$+\mathrm{cr}_D(K_{5,n-2},T^k\cup T^l)+\mathrm{cr}_D(G,T^k\cup T^l)\geq$$
$$\geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+n-2+\left\lfloor\frac{n-2}{2}\right\rfloor+0+$$
$$+4(n-2)+3=4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor.$$

This contradiction confirms that D is antipodal-free. Without lost of generality, we can assume the configuration A_3 of the subgraph $F^n = G \cup T^n$ for $T^n \in S_D$. It is obvious, if there is a subgraph $T^i \notin S_D$ with $\mathrm{cr}_D(G \cup T^n, T^i) = 3$, then $\mathrm{cr}_D(G, T^i) = 2$ and $\mathrm{cr}_D(T^n, T^i) = 1$. Thus, by a discussion of replacement in possible regions in the subdrawing of $G \cup T^n$, the vertex t_i must be placed in the triangular region with three vertices of G on its boundary, i.e. there is only one possibility in such a way as shown in Fig. 4.

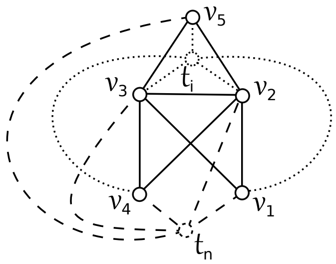

Fig. 4 Drawing in which $\mathrm{cr}_D(G \cup T^n, T^i) = 3$ for $T^n \in S_D$ and $T^i \notin R_D \cup S_D$

Let T^i be the considered subgraph as in Fig. 4. Since the subgraph F^i is represented by the cyclic permutation (15432) and the minimum number of interchanges of adjacent elements of $\overline{(15432)}$ required to produce cyclic permutation $\overline{(14325)} = (15234)$ or $\overline{(14532)} = (12354)$ is three, then any subgraph T^j with the configuration A_1 or A_2 of F^j crosses the edges of T^i at least thrice, respectively. So, $\mathrm{cr}_D(T^n \cup T^i, T^k) \geq 2+3 = 4+1 = 5$ for any $T^k \in S_D$ with $k \neq n$ by Table 1, and $\mathrm{cr}_D(T^n \cup T^i, T^k) \geq 3$ for any $T^k \notin S_D$ with $k \neq i$. Thus, by fixing the graph $T^n \cup T^i$,

$$\mathrm{cr}_D(G+D_n) \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+n-2+\left\lfloor\frac{n-2}{2}\right\rfloor+$$
$$+5(s-1)+3(n-s-1)+1+3 \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+$$
$$+\left\lfloor\frac{n-2}{2}\right\rfloor+4n+2\left(n-\left\lfloor\frac{n}{2}\right\rfloor+2\right)-6 \geq$$
$$\geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor.$$

In addition, let us assume that there is no $T^i \notin S_D$ with $\mathrm{cr}_D(G \cup T^n, T^i) = 3$. Let us denote $S_D(A_1,A_2) = \{T^i \in S_D : F^i$ having configuration A_1 or $A_2\}$, and $S_D(A_3,A_4) = \{T^i \in S_D : F^i$ having configuration A_3 or $A_4\}$. Note that $S_D(A_1,A_2)$ and $S_D(A_3,A_4)$ are disjoint subsets of S_D. Hence, if we denote by $s_1 = |S_D(A_1,A_2)|$ and $s_2 = |S_D(A_3,A_4)|$, then $s_1+s_2 = s$. Moreover, $s_2 \geq 1$ by our assumption. Thus, we will discuss two cases in which are used the values in Table 1.:

1. Suppose that $s_1 \leq \left\lfloor\frac{n}{2}\right\rfloor$, that is, $-s_1 \geq -\left\lfloor\frac{n}{2}\right\rfloor$. By fixing the graph $G \cup T^n$,

$$\mathrm{cr}_D(G+D_n) \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+3s_1+$$
$$+5(s_2-1)+4(n-s_1-s_2)+1+1 =$$
$$= 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+4n+s_2-s_1-3 \geq$$
$$\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+4n+1-\left\lfloor\frac{n}{2}\right\rfloor-3 \geq$$
$$\geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor.$$

2. Suppose that $s_1 > \left\lfloor\frac{n}{2}\right\rfloor$, that is, $s_1 \geq \left\lfloor\frac{n}{2}\right\rfloor+1$. Since $s_1 = |S_D(A_1,A_2)|$, there are at least $\left\lceil\frac{s_1}{2}\right\rceil$ different subgraphs T^i with the same configuration $A_k \in \mathscr{M}_D$ of the subgraphs F^i, for some $k \in \{1,2\}$. Hence, by fixing the graph T^i,

$$\mathrm{cr}_D(G+D_n) \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+n-1+$$
$$+\left\lfloor\frac{n-1}{2}\right\rfloor+4\left(\left\lceil\frac{s_1}{2}\right\rceil-1\right)+2\left\lfloor\frac{s_1}{2}\right\rfloor+2s_2+$$
$$+1(n-s_1-s_2)+1 = 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+2n+$$
$$+\left\lfloor\frac{n-1}{2}\right\rfloor+2\left\lceil\frac{s_1}{2}\right\rceil+s-4 \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+$$
$$+2n+\left\lfloor\frac{n-1}{2}\right\rfloor+s_1+\left(n-\left\lfloor\frac{n}{2}\right\rfloor+2\right)-4 \geq$$
$$\geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+3n+\left\lfloor\frac{n-1}{2}\right\rfloor-\left\lfloor\frac{n}{2}\right\rfloor+$$
$$+\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)-2 \geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor.$$

Due to symmetry, the same arguments are applied for the case $A_4 \in \mathscr{M}_D$.

(b) $\mathscr{M}_D = \{A_1,A_2\}$. Without lost of generality, let us consider two different subgraphs T^n, $T^{n-1} \in S_D$ such that F^n and F^{n-1} have configurations A_1 and A_2, respectively. As $\mathscr{M}_D = \{A_1,A_2\}$, we have $\mathrm{cr}_D(T^n \cup T^{n-1}, T^i) \geq 6$ for any $T^i \in S_D$ with $i \neq n-1, n$. Then, by fixing the graph $T^n \cup T^{n-1}$,

$$\mathrm{cr}_D(G+D_n) \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+n-2+\left\lfloor\frac{n-2}{2}\right\rfloor+$$
$$+6(s-2)+2(n-s)+2+2 \geq 4\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor+$$
$$+\left\lfloor\frac{n-2}{2}\right\rfloor+3n+4\left(n-\left\lfloor\frac{n}{2}\right\rfloor+2\right)-10 \geq$$
$$\geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor.$$

(c) $\mathscr{M}_D = \{A_j\}$ for only one $j \in \{1,2\}$. Without lost of generality, we can assume that the configuration of F^n is A_1. By fixing the graph T^n,

$$\mathrm{cr}_D(G+D_n) \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+n-1+\left\lfloor\frac{n-1}{2}\right\rfloor+$$
$$+4(s-1)+1 \geq 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor+\left\lfloor\frac{n-1}{2}\right\rfloor+n+$$
$$+4\left(n-\left\lfloor\frac{n}{2}\right\rfloor+2\right)-4 \geq 4\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor+n+\left\lfloor\frac{n}{2}\right\rfloor.$$

Due to symmetry, the same arguments are used for the case $\mathscr{M}_D = \{A_2\}$.

Case 2: $\text{cr}_D(G) = 3$. There is one subdrawing of G with three crossings among its edges shown in Fig. 1(f). Without lost of generality, we can choose the vertex notation of the graph G as shown in Fig. 5(a).

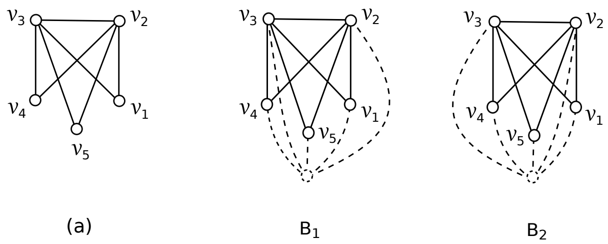

Fig. 5 Two drawings of the subgraph $F^i = G \cup T^i$, where $T^i \in S_D$

The reader can easy to verify that if there is a $T^i \in R_D$, then the subgraph F^i is represented by $\text{rot}_D(t_i) = (15432)$. Hence, we can use the same idea as in Case 1. Our aim is to list all possible rotations $\text{rot}_D(t_i)$ which can appear in D if the edges of T^i cross the edges of G exactly once. Of course, the edges t_iv_1, t_iv_4 and t_iv_5 can not cross the edges of graph G in the considered subgraph T^i. Thus, we only need to consider two drawings of the subgraph F^i denoted as the configurations B_1 and B_2, see Fig. 5. By condition (1), if there is no $T^i \in R_D$ then the set S_D is nonempty. Moreover, we are able to inspect by a discussion that if $T^i \in S_D$ with the configuration B_k of F^i for some $k \in \{1,2\}$, then there is no subgraph $T^j \notin R_D$ with $\text{cr}_D(G \cup T^i, T^j) \leq 3$ for $j \neq i$. Hence, we can also use the same idea as in Case 1.

Thus, it was shown that there is no good drawing D of the graph $G + D_n$ with less than $4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor$ crossings. This completes the proof of the main theorem.

4. COROLLARY

Let H be the graph obtained from G by adding the edge v_1v_4 in the subdrawing in Fig. 1(b).

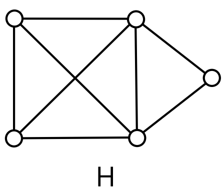

Fig. 6 The graph H by adding one edge to the graph G

Since we are able to add this edge to the graph G without additional crossings in Fig. 3(b), the drawing of the graph $H + D_n$ with $4\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + n + \lfloor \frac{n}{2} \rfloor$ crossings is obtained. Thus, the next result is obvious.

Corollary 4.1. $\text{cr}(H + D_n) = 4\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + n + \left\lfloor \frac{n}{2} \right\rfloor$ *for any* $n \geq 1$.

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