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THE CROSSING NUMBER OF $P_N^2 \square C_4$

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ABSTRACT

The exact crossing number is known only for few specific families of graphs. According to their special structure, Cartesian products of two graphs are one of few graph classes for which the exact values of crossing numbers were obtained. Let P_n be a path with $n+1$ vertices and P_n^k be the k-power of the graph P_n. Very recently, some results concerning crossing numbers of P_n^k were obtained. For the Cartesian product of P_n^2 with the cycle of length three, the value $3n-3$ for its crossing number is given. In this paper, we extend this result by proving that the crossing numbers of the Cartesian product $P_n^2 \square C_4$ is $4n-4$.

Keywords: *graph, drawing, crossing number, Cartesian product, path, cycle*

1. INTRODUCTION

Let G be a simple graph with vertex set V and edge set E. A *drawing* of a graph is a mapping of a graph into a surface. For simplicity, we assume that in a drawing (a) no edge passes through any vertex other than its end-points, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross), and (c) no three edges cross at the same point. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross. The *crossing number* $\mathrm{cr}(G)$ of a graph G is the number of edge crossings in an optimal drawing of G in the plane. The *Cartesian product* $G_1 \square G_2$ of two graphs G_1 and G_2 has vertices $V(G_1) \times V(G_2)$ and two vertices (u,u') and (v,v') are adjacent in $G_1 \square G_2$ if and only if $u = v$ and $u'v' \in E(G_2)$ or $u' = v'$ and $uv \in E(G_1)$.

Let C_n be the cycle on n vertices and P_n be the path on $n+1$ vertices. Harary et al. [7] conjectured that $\mathrm{cr}(C_m \square C_n) = (m-2)n$, for all m,n satisfying $3 \leq m \leq n$. This has been proved only for m,n satisfying $n \geq m$, $m \leq 7$. It was proved by Glebsky and Salazar [6] that $\mathrm{cr}(C_m \square C_n)$ equals its long-conjectured value at least for $n \geq m(m+1)$. Besides of Cartesian products of two cycles, there are several other exact results. Beineke and Ringeisen [1, 19] started to study the crossing numbers of Cartesian products of graphs of order at most four with paths and cycles. The crossing numbers of $G \square C_n$ for all graphs G of order at most four are given in [8, 9]. In addition, the crossing numbers of $G \square C_n$ are known for some graphs G on five or six vertices [4, 5, 14]. In [8, 12] the crossing numbers of $G \square P_n$ for all graphs on at most five vertices are established and the crossing numbers of Cartesian products of graphs of order five or six with stars were studied in [2, 10, 11].

For any positive integer k, the *k-power graph* of a graph G, denoted by G^k, is the graph having the same vertices as G, and two vertices of G^k are adjacent if the distance between the corresponding vertices in G is at most k. In the paper [18], Patil and Krishnnamurthy present a family of graphs for which power graphs have crossing number one. The crossing numbers of Cartesian products for some second power P_n^2 of the path P_n with paths and cycles are determined in [15–17]. Specifically, it is proved that the crossing number of the graph $P_n^2 \square C_3$ is $3n-3$ for all $n \geq 2$. In this paper, we extend this result by giving the crossing number of the graph $P_n^2 \square C_4$.

2. THE GRAPH $P_5^2 \square C_4$

Assume $n \geq 2$. We find it convenient to regard the graph $P_n^2 \square C_4$ in the following way: it has $4(n+1)$ vertices and edges that are the edges in $n+1$ copies $C_4^{(i)}$, $i = 0,1,\ldots,n$, of the cycle C_4 and in four copies of P_n^2.

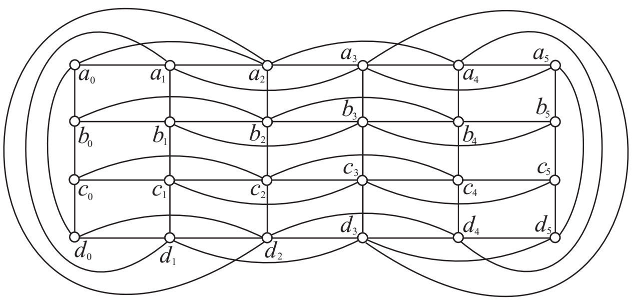

Fig. 1 The graph $P_5^2 \square C_4$ with sixteen crossings

In Fig. 1 there is the drawing of the graph $P_5^2 \square C_4$, in which every cycle $C_4^{(i)}$ with vertices a_i, b_i, c_i, d_i, $i = 1,2,3,4$, is crossed exactly four times and there are no other crossings in this drawing. So, Fig. 1 shows the drawing of the graph $P_5^2 \square C_4$ with sixteen crossings. The extension of this drawing to the drawing of $P_n^2 \square C_4$ with four crossings on the edges of every cycle $C_4^{(i)}$, $i = 1,2,\ldots,n-1$, implies that the crossing number of the graph $P_n^2 \square C_4$ is at most $4(n-1)$.

In the next section we will discuss some properties of drawings of the graph $P_5^2 \square C_4$ pertaining some special subgraphs of it. Using these properties, in the last section of the paper we give the exact value of the crossing number for the graph $P_n^2 \square C_4$.

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3. DRAWINGS OF THE GRAPH $P_N^2 \square C_4$

In the graph $P_n^2 \square C_4$, every of four copies of P_n^2 is induced by the vertices x_i, $i = 0,1,\dots,n$, for some $x \in \{a,b,c,d\}$. For $i = 1,2,\dots,n$, let $P^{(i)}$ denote the subgraph of $P_n^2 \square C_4$ consisting of the edges $a_{i-1}a_i$, $b_{i-1}b_i$, $c_{i-1}c_i$, and d_{i-1}, d_i joining the cycles $C_4^{(i-1)}$ and $C_4^{(i)}$. Similarly, for $i = 1,2,\dots,n-1$, let $H^{(i)}$ denote the subgraph of $P_n^2 \square C_4$ consisting of the edges $a_{i-1}a_{i+1}$, $b_{i-1}b_{i+1}$, $c_{i-1}c_{i+1}$, and $d_{i-1}d_{i+1}$ joining the cycles $C_4^{(i-1)}$ and $C_4^{(i+1)}$. Thus, the subgraph $P^{(1)} \cup P^{(2)} \cup \cdots \cup P^{(n)}$ consists of four copies of P_n, and two vertices of P_n at distance two are adjacent with an edge of some $H^{(i)}$, see the drawing of $P_5^2 \square C_4$ in Fig. 1. For $i = 0,1,\dots,n-2$, let $G^{i,i+2}$ denote the subgraph of the graph $P_n^2 \square C_4$ induced by the vertices of the cycles $C_4^{(i)}$, $C_4^{(i+1)}$, and $C_4^{(i+2)}$. Clearly, every subgraph $G^{i,i+2}$ is isomorphic to the Cartesian product $C_3 \square C_4$. Especially, the subgraph $G^{0,2} = C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)} \cup P^{(2)} \cup C_4^{(2)} \cup H^{(1)}$. For $x \in \{a,b,c,d\}$, let $C_3^{(x)}$ be the 3-cycle of the subgraph $G^{0,2}$ induced by the vertices x_0, x_1, and x_2.

Let D be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. For a subgraph G_i of the graph G, let $D(G_i)$ be the subdrawing of G_i induced by D. For edge disjoint subgraphs G_i and G_j of G, we denote by $\mathrm{cr}_D(G_i, G_j)$ the number of crossings of edges in G_i with edges in G_j, and by $\mathrm{cr}_D(G_i)$ the number of crossings among edges of G_i in D. In a good drawing D of the graph G, we say that a cycle C *separates* the cycles C' and C'' (the vertices of a subgraph G_i not containing vertices of C) if C' and C'' (the vertices of G_i) are contained in different components of $\mathbb{R}^2 \setminus C$.

Let D be a good drawing of the graph $P_n^2 \square C_4$. Consider the subgraphs $C_4^{(0)} \cup P^{(1)} \cup H^{(1)}$ and $C_4^{(1)} \cup P^{(2)}$ of the graph $P_n^2 \square C_4$ drawn by solid lines in Fig. 2(a) and Fig. 2(b), respectively. Let us call the edges of $C_4^{(0)} \cup P^{(1)} \cup H^{(1)}$ red and let the edges of $C_4^{(1)} \cup P^{(2)}$ be called blue. We say that a crossing in D is *red* if it involves at least one edge of $C_4^{(0)} \cup P^{(1)} \cup H^{(1)}$. Similarly, let a crossing on an edge of $C_4^{(1)} \cup P^{(2)}$ be *blue*. Clearly, a crossing between an edge of $C_4^{(0)} \cup P^{(1)} \cup H^{(1)}$ and an edge of $C_4^{(1)} \cup P^{(2)}$ is both red and blue. Such a crossing we call *red-blue*. Let α and β denote the numbers of red and blue crossings, respectively, and let (α, β) be the number of red-blue crossings. Let us dissect the numbers α and β into two parts. Namely, let $\alpha = \alpha^i + \alpha^e$ and $\beta = \beta^i + \beta^e$, where α^i and β^i be the numbers of red and blue crossings, respectively, which appear among the edges of the subgraph $G^{0,2}$. Clearly, α^e and β^e are the numbers of crossings between red and blue edges of $G^{0,2}$, respectively, and edges not belonging to $G^{0,2}$.

The edges of a 4-cycle cross each other at most once in a good drawing. Except for a possible crossing among the edges of $C_4^{(2)}$, in D, every other crossing among the edges of $G^{0,2}$ appears on a red or on a blue edge. So, $\alpha + \beta \geq 3$, because the crossing number of the graph $C_3 \square C_4$ is four, see [1]. For the considered drawing D, several propositions hold.

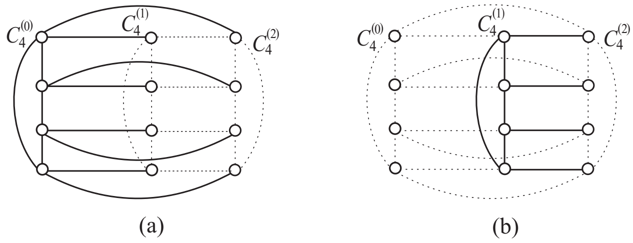

Fig. 2 The subgraphs $C_4^{(0)} \cup P^{(1)} \cup H^{(1)}$ and $C_4^{(1)} \cup P^{(2)}$

Proposition 3.1. *For $i = 0,1,2,\dots,n-1$, $\mathrm{cr}_D(C_4^{(i)} \cup P^{(i+1)} \cup C_4^{(i+1)}) \neq 1$, as well as $\mathrm{cr}_D(C_4^{(i)} \cup H^{(i+1)} \cup C_4^{(i+2)}) \neq 1$ for all $i = 0,1,2,\dots,n-2$.*

Proof. Both subgraphs $C_4^{(i)} \cup P^{(i+1)} \cup C_4^{(i+1)}$ and $C_4^{(i)} \cup H^{(i+1)} \cup C_4^{(i+2)}$ of the graph $P_n^2 \square C_4$ are isomorphic to the Cartesian product $P_1 \square C_4$. This graph is planar. For any graph $P_1 \square C_n$, $n \geq 4$, there is no good drawing with exactly one crossing, because for any two edges which cross each other one can find two vertex-disjoint cycles such that crossed edges are in different cycles. Two vertex-disjoint cycles cannot cross only once. □

If, in D, the edges of some 4-cycle of the graph $P_5^2 \square C_4$ cross each other, we say that this 4-cycle has an *internal* crossing. As the drawing D is good, the edges of a 4-cycle cannot cross more than once, and the edges of every 3-cycle are pairwise non-crossing.

Proposition 3.2. *Let p be the number of the 4-cycles $C_4^{(i)}$, $i \in \{0,1,2\}$, with internal crossings and q be the number of crossings on the edges of a 3-cycle $C_3^{(x)}$, $x \in \{a,b,c,d\}$. Then there are at least $3+p+q$ crossings among the edges of the subgraph $G^{0,2}$.*

Proof. The subgraph $G^{0,2}$ of the graph $P_n^2 \square C_4$ is isomorphic to $C_3 \square C_4$. Let $D(G^{0,2})$ be the subdrawing of $G^{0,2}$ induced by D. By the removal of all edges of some 3-cycle $C_3^{(x)}$, $x \in \{a,b,c,d\}$, with q crossings on its edges, a subdivision of the graph $C_3 \square C_3$ is obtained, and the resulting subdrawing has at least q crossings less than the drawing $D(G^{0,2})$. As $\mathrm{cr}(C_3 \square C_3) = 3$, see [19], $D(G^{0,2})$ must contain at least $3+q$ crossings. Moreover, no 3-cycle has an internal crossing in a good drawing of a graph. Assume that some 4-cycle $C_4^{(i)}$, $i \in \{0,1,2\}$, has an internal crossing in $D(G^{0,2})$. Then in the subdrawing of $D(G^{0,2})$ obtained by deleting the edges of some $C_3^{(x)}$, $x \in \{a,b,c,d\}$, the edges of the corresponding 4-cycle $C_4^{(i)}$ in the subdivision of $C_3 \square C_3$ cross each other. But, this subdrawing is not optimal, because the edges of a 3-cycle do not cross each other in any optimal drawing of $C_3 \square C_3$. This forces that the original drawing $D(G^{0,2})$ has at least one crossing more than the minimum number of crossing in a drawing of the graph $C_3 \square C_3$. If p 4-cycles have internal crossings in $D(^{0,2}$, by deleting one of them the subdrawing of a subdivision of $C_3 \square C_3$ with at least one crossing less is obtained. Moreover, $p-1$ 3-cycles have internal crossings in this sudrawing. These crossings are not necessary in an

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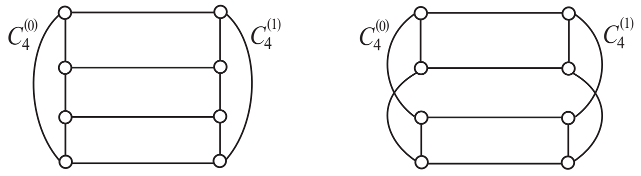
optimal drawing of $C_3 \Box C_3$. Thus, if there are p 4-cycles $C_4^{(i)}$, $i \in \{0,1,2\}$, with internal crossings and some 3-cycle $C_3^{(x)}$, $x \in \{a,b,c,d\}$, has q crossings on its edges, then there are at least $3+p+q$ crossings in the subdrawing $D(G^{0,2})$ induced by D. □

Proposition 3.3. *If two different 3-cycles $C_3^{(x)}$ and $C_3^{(y)}$, $x,y \in \{a,b,c,d\}$, of the subgraph $G^{0,2}$ cross each other in D, then at least one of them has at least three crossings on its edges. If two 4-cycles $C_4^{(i)}$ and $C_4^{(i+1)}$, $i = 0,1,\ldots,n-1$, cross each other in D, then at least one of them has at least three crossings on its edges.*

Proof. Every two 4-cycles $C_4^{(i)}$ and $C_4^{(i+1)}$ belong to some subgraph of $P_n^2 \Box C_4$ isomorphic to $C_3 \Box C_4$. Without loss of generality, consider the subgraph $G^{0,2}$ isomorphic with $C_3 \Box C_4$. If two 4-cycles cross each other in the good subdrawing $D(G^{0,2})$ of $G^{0,2}$, then they cross at least twice. Assume two 4-cycles which are crossing each other and none of them has more than two crossings. Then none of them has an internal crossing. The subdrawing of $D(G^{0,2})$ induced by the edges of the considered two 4-cycles divides the plane into four regions. Since, in $D(G^{0,2})$, none of the other edges of $C_3 \Box C_4$ crosses an edge of the considered two 4-cycles, all other edges of $G^{0,2}$ are placed in $D(G^{0,2})$ within one region formed by these two 4-cycles and such a drawing is not good. The same arguments we can use for the 3–cycles of $G^{0,2}$. This completes the proof. □

Proposition 3.4. *If $\mathrm{cr}_D(C_4^{(0)}, C_4^{(1)}) = 0$ and every 3-cycle $C_3^{(x)}$ of the subgraph $G^{0,2}$ has at most one crossing on its edges, then $\alpha > 3$ or $\beta > 3$.*

Proof. Consider the 3-cycles in $G^{0,2}$. By Proposition 3.3, $\mathrm{cr}_D(C_3^{(x)}, C_3^{(y)}) = 0$ for all pairs $x,y \in \{a,b,c,d\}$. Moreover, none of the 3-cycles separates two other, otherwise it is crossed by all three edges joining the separated 3-cycles. Assume, without loss of generality, that some 3-cycle $C_3^{(x)}$ crosses the 4-cycle $C_4^{(0)}$ in D. If $C_4^{(0)}$ is crossed once by some edge of $P^{(2)}$, then $C_4^{(0)}$ separates $C_4^{(1)}$ and $C_4^{(2)}$ or $C_4^{(0)}$ is crossed by $C_4^{(2)}$ twice. In the first case, $C_4^{(0)}$ is crossed by all four edges of $P^{(2)}$ and $(\alpha,\beta) \geq 4$. In the second case, $C_4^{(0)}$ is crossed by $P^{(2)} \cup C_4^{(2)}$ at least three times and also by the path $x_2x_3x_1$ joining the end-vertices x_1 and x_2 of the edge, which crosses $C_4^{(0)}$. Thus, $\alpha \geq 4$. If $C_4^{(0)}$ is crossed by an edge of $P^{(1)}$ ($H^{(1)}$) belonging to $C_3^{(x)}$, then, as $C_3^{(x)}$ does not separate two other 3-cycles and two 3-cycles do not cross each other, the edge of $H^{(1)}$ ($P^{(1)}$) belonging to $C_3^{(x)}$ crosses the same edge of $C_4^{(0)}$, too. Hence, $C_3^{(0)}$ cannot be crossed by some 3-cycle of $G^{0,2}$. The same fact can be shown for the 4-cycle $C_4^{(1)}$. This, together with the restriction $\mathrm{cr}_D(C_4^{(0)}, C_4^{(1)}) = 0$ implies that there are only two possible subdrawings $D(C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)})$ induced by D. Both are shown in Fig. 3.



(a) (b)

Fig. 3 The subdrawings of $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$

Assume first the subdrawing of $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$ in Fig. 3(a). If $C_4^{(0)}$ separates the 4-cycles $C_4^{(2)}$ and $C_4^{(1)}$, then $C_4^{(0)}$ is crossed by all four edges of $P^{(2)}$ and $(\alpha,\beta) \geq 4$. Similarly, $(\alpha,\beta) \geq 4$, if $C_4^{(1)}$ separates $C_4^{(2)}$ and $C_4^{(0)}$. If the 4-cycle $C_4^{(0)}$ separates two adjoint vertices x_2 and y_2 of $C_4^{(2)}$, then $\mathrm{cr}_D(C_4^{(0)}, C_4^{(2)}) \geq 2$ and at least one edge of $P^{(2)}$ joining $C_4^{(2)}$ with $C_4^{(1)}$ crosses $C_4^{(0)}$, too. Moreover, in D, the path $x_2x_3y_3y_2$ crosses $C_4^{(0)}$ and therefore, $\alpha \geq 4$. The same analysis for the case when $C_4^{(1)}$ separates two vertices of $C_4^{(2)}$ shows that $\beta \geq 4$. Hence, if both $\alpha \leq 3$ and $\beta \leq 3$, the whole 4-cycle $C_4^{(2)}$ is placed in D outside the 4-cycles $C_4^{(0)}$ and $C_4^{(1)}$ in the view of the subdrawing shown in Fig. 3(a). If all vertices of $C_4^{(2)}$ are placed in one quadrangular region or in two neighbouring quadrangular regions of $D(C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)})$, then at least two vertices x_0 and x_1 are not placed on the boundaries of these considered regions. But, in this case, the 3-cycle $C_3^{(x)}$ crosses the edges of $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$ at least twice, a contradiction. The last possibility is that the vertices of $C_4^{(2)}$ are placed in at least two quadrangular regions of $D(C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)})$ outside $C_4^{(0)}$ and $C_4^{(1)}$ and two of these regions are not neighbouring regions. Thus, the cycle $C_4^{(2)}$ crosses the edges of $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$ at least four times. If $\mathrm{cr}_D(C_4^{(1)}, C_4^{(2)}) = 0$, then $\alpha \geq 4$. Otherwise $\mathrm{cr}_D(C_4^{(1)}, C_4^{(2)}) \geq 2$ and $\alpha^i + \beta^i \geq 4$.

Consider now the subgraph $C_4^{(3)} \cup P^{(4)} \cup C_4^{(4)}$. If this 3-connected graph crosses the 3-connected graph $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$, then it crosses it at least three times and $\alpha^e + \beta^e \geq 3$, which forces $\alpha \geq 4$ or $\beta \geq 4$. Otherwise it is placed in one region of $D(C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)})$. If no vertex of $C_4^{(2)}$ is placed in the same region, then all four edges of $P^{(3)}$ joining $C_4^{(3)}$ with $C_4^{(2)}$ cross the edges of $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$ and $\alpha^e + \beta^e \geq 4$. If $C_4^{(3)} \cup P^{(4)} \cup C_4^{(4)}$ is placed in a region with a vertex of $C_4^{(2)}$, then at least one vertex of $C_4^{(2)}$ and at least two vertices of $C_4^{(1)}$ are not placed in this region or on its boundary. In this case, at least two paths $x_2x_3y_3y_2$ and $x_2x_4y_4y_2$ joining two separated vertices x_2 and y_2 of $C_4^{(2)}$ cross the edges of $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$. Moreover, the edges of $H^{(2)}$ joining $C_4^{(3)}$ with $C_4^{(1)}$ cross the edges of $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$. Thus, $\alpha^e + \beta^e \geq 3$ again. This confirms that $\alpha + \beta \geq 7$ in all possible cases and therefore, $\alpha \geq 4$ or $\beta \geq 4$.

In the subdrawing of $C_4^{(0)} \cup P^{(1)} \cup C_4^{(1)}$ in Fig. 3(b) there

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is one red crossing and one blue crossing. The same analysis as in the previous case states that $\alpha \geq 4$ or $\beta \geq 4$. This completes the proof. □

4. THE MAIN RESULT

Theorem 4.1. $\mathrm{cr}(P_n^2 \Box C_4) = 4(n-1)$ *for* $n \geq 2$.

Proof. In Fig. 1 it is easy to see that $\mathrm{cr}(P_n^2 \Box C_4) \leq 4(n-1)$ for $n \geq 2$. As P_2^2 is isomorphic to the cycle C_3 and $\mathrm{cr}(C_3 \Box C_m) = m$ [19], the crossing number of the graph $P_2^2 \Box C_4$ is four. The graph P_3^2 is isomorphic to the tripartite complete graph $K_{1,1,2}$ and contains C_4 as a subgraph. So, the graph $P_3^2 \Box C_4$ contains $C_4 \Box C_4$ as a subgraph and, as $\mathrm{cr}(C_4 \Box C_4) = 8$ [3], the crossing number of the graph $P_3^2 \Box C_4$ is at least eight. Hence, $\mathrm{cr}(P_3^2 \Box C_4) = 8$ and the result is true for $n = 3$. In [13], some crossing numbers of Cartesian products of graphs on five vertices with cycles are collected. For the graph P_4^2 of order five, it is shown that $\mathrm{cr}(P_4^2 \Box C_m) = 3m$. This implies that $\mathrm{cr}(P_4^2 \Box C_4) = 12$. The crossing number of the graph $P_5^2 \Box C_m$ is $4m$, see [15], and therefore, $\mathrm{cr}(P_5^2 \Box C_4) = 16$. It remains to show that in any good drawing of the graph $P_n^2 \Box C_4$, $n \geq 6$, there are at least $4(n-1)$ crossings. We prove this by induction on n.

Assume that for $n \geq 6$ there is a good drawing D of the graph $P_n^2 \Box C_4$ with less than $4n-4$ crossings and let $\mathrm{cr}(P_k^2 \Box C_4) = 4k-4$ for every $2 \leq k \leq n-1$. Hence, in D, there are at most three crossings on red edges of $C_4^{(0)} \cup P^{(1)} \cup H^{(1)}$ and at most three crossings on blue edges of $C_4^{(1)} \cup P^{(2)}$. Otherwise, by deleting the edges of $C_4^{(0)} \cup P^{(1)} \cup H^{(1)}$ from D, a drawing of the graph $P_{n-1}^2 \Box C_4$ with fewer than $4(n-1)-4$ crossings is obtained. Similarly, the removing of the edges of $C_4^{(1)} \cup P^{(2)}$ results in a drawing of the subdivision of $P_{n-1}^2 \Box C_4$ with fewer than $4(n-1)-4$ crossings. This contradicts the induction hypothesis and therefore, $\alpha \leq 3$ and $\beta \leq 3$ in D.

Consider now the subgraph $G^{0,2} = C_4^{(0)} \cup P^{(1)} \cup H^{(1)} \cup C_4^{(1)} \cup P^{(2)} \cup C_4^{(2)}$ and let $D(G^{0,2})$ be its subdrawing induced by D. As $\alpha \leq 3$, at least one crossing appears among the edges of $C_4^{(1)} \cup P^{(2)} \cup C_4^{(2)}$. Thus, by Proposition 3.1 we have that $\mathrm{cr}_D(C_4^{(1)} \cup P^{(2)} \cup C_4^{(2)}) \geq 2$. This implies that in D there are at most two red-blue crossings. We show that the 4-cycles $C_4^{(0)}$ and $C_4^{(1)}$ do not cross each other. Otherwise, if $\mathrm{cr}_D(C_4^{(0)}, C_4^{(1)}) \neq 0$, then both crossings between $C_4^{(0)}$ and $C_4^{(1)}$ are red-blue crossings and at least one of the considered 4-cycles separates the vertices of the other. Assume first that $C_4^{(0)}$ separates the vertices x_1 and y_1 of $C_4^{(1)}$. As $\alpha \leq 3$, the cycle $C_4^{(2)}$ cannot cross $C_4^{(0)}$. Then the path $x_1x_2y_2y_1$ crosses the cycle $C_4^{(0)}$ in such a way that $C_4^{(0)}$ is crossed by the edge x_1x_2 or by the edge y_1y_2. This crossing is also a red-blue crossing, a contradiction with our observation above. The same analysis for the case when the cycle $C_4^{(1)}$ separates the vertices of $C_4^{(0)}$ confirms that $\mathrm{cr}_D(C_4^{(0)}, C_4^{(1)}) = 0$.

By Proposition 3.4, at least one 3-cycle $C_3^{(x)}$, $x \in \{a,b,c,d\}$, has more than one crossing on its edges in $D(G^{0,2})$. Thus, by Proposition 3.2, in $D(G^{0,2})$ there are at least five crossings when the edges of $C_4^{(2)}$ do not cross each other, and at least six crossings when $\mathrm{cr}_D(C_4^{(2)}) \neq 0$. The restriction $\alpha + \beta \leq 6$ implies that in the first case $5 \leq \mathrm{cr}_D(G^{0,2}) \leq 6$, and $6 \leq \mathrm{cr}_D(G^{0,2}) \leq 7$ in the second case. This forces that, in D, at most one of the numbers (α, β) and $\alpha^e + \beta^e$ is equal to one, and the other is zero. We show that this is impossible.

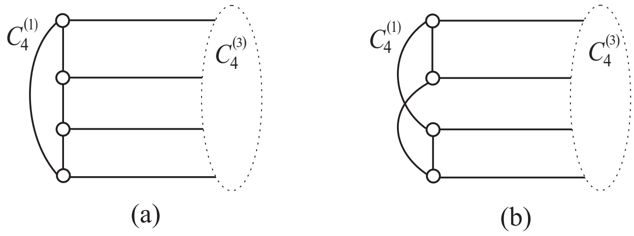

Fig. 4 The subdrawings of $C_4^{(1)} \cup H^{(2)} \cup C_4^{(3)}$

If $\beta^e = 0$, then only the edges of the subgraph $G^{0,2}$ can cross the edge of $C_4^{(1)}$ and the subdrawing of $C_4^{(1)} \cup H^{(2)} \cup C_4^{(3)}$ induced by D divides the plane in such a way that on the boundary of every region there are at most two vertices of $C_4^{(1)}$. According to the numbers of crossings among the edges of $C_4^{(1)}$, Fig. 4 shows both subdrawings $D(C_4^{(1)} \cup H^{(2)} \cup C_4^{(3)})$, where possible crossings among the edges of $H^{(2)} \cup C_4^{(3)}$ are considered in the dotted cycle. As $\mathrm{cr}_D(C_4^{(0)}, C_4^{(1)}) = 0$ and $\alpha^e \leq 1$, $\mathrm{cr}_D(C_4^{(0)}, C_4^{(1)} \cup H^{(2)} \cup C_4^{(3)}) = 0$ and, in D, the cycle $C_4^{(0)}$ is placed in one region in the view of the subdrawing $D(C_4^{(1)} \cup H^{(2)} \cup C_4^{(3)})$. No vertex of $C_4^{(2)}$ can be placed in the region with all four vertices of $C_4^{(1)}$ on its boundary, otherwise some edge of $P^{(3)}$ crosses $C_4^{(1)}$ and $\beta^e \neq 0$. This implies that $C_4^{(0)}$ is also outside $C_4^{(1)}$, otherwise all four edges of $H^{(1)}$ cross $C_4^{(1)}$ and $\beta \geq 4$. Now, as $C_4^{(0)}$ is placed in a region with at most two vertices of $C_4^{(1)}$ on its boundary, at least two edges of $P^{(1)}$ joining $C_4^{(0)}$ with $C_4^{(1)}$ cross the edges of $C_4^{(1)} \cup H^{(2)} \cup C_4^{(3)}$. But, if an edge of $P^{(1)}$ crosses $C_4^{(1)}$, then $(\alpha, \beta) = 1$. The restriction of $(\alpha, \beta) \leq 1$ forces that $\alpha^e = 0$, which is impossible, because the other edge of $P^{(1)}$ crosses an edge of $H^{(2)} \cup C_4^{(3)}$, and this crossing contributes at least one to α^e.

Hence, $\beta^e = 1$ and $\alpha^e = 0$ in the drawing D of the graph $P_n^2 \Box C_4$. Moreover, in D, there is no red-blue crossing and $\alpha^i + \beta^i \leq 5$. This implies that $\mathrm{cr}_D(G^{0,2}) = 5$ if $\mathrm{cr}_D(C_4^{(2)}) = 0$, and for the case $\mathrm{cr}_D(C_4^{(2)}) = 1$ we have $\mathrm{cr}_D(G^{0,2}) = 6$. Thus, by Proposition 3.2, every 3-cycle $C_3^{(x)}$ of $G^{0,2}$ has at most two crossings on its edges and therefore, no 3-cycle of $G^{0,2}$ separates two other 3-cycles. In addition, by Proposition 3.3, two different 3-cycles $C_3^{(x)}$ and $C_3^{(y)}$ does not cross each other. It was discussed above that, by these restrictions, two red edges of $H^{(1)}$ do not cross each other and no edge of $H^{(1)}$ crosses $C_4^{(0)}$. Hence, according to the number of internal crossings in $C_4^{(0)}$, all subdrawings of $C_4^{(0)} \cup H^{(1)} \cup P^{(3)} \cup C_4^{(3)}$ induced by D are shown in Fig. 5,

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where possible crossings among the edges of $P^{(3)} \cup C_4^{(3)}$ are considered in the dotted cycles.

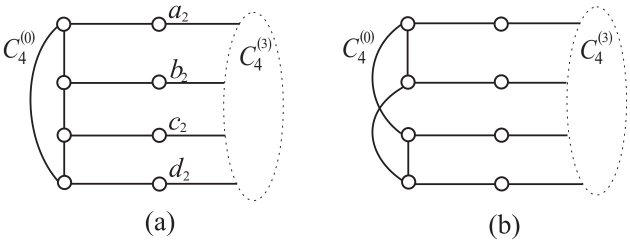

Fig. 5 The subdrawings of $C_4^{(0)} \cup H^{(1)} \cup P^{(3)} \cup C_4^{(3)}$

As $(\alpha,\beta) = 0$ and $\beta^e = 1$, the cycle $C_4^{(1)}$ does not cross the edges of $C_4^{(0)} \cup H^{(1)}$ or the edges of $P^{(3)} \cup C_4^{(3)}$ twice. So, $C_4^{(1)}$ is placed in D in one region in the view of the subdrawing $D(C_4^{(0)} \cup H^{(1)} \cup P^{(3)} \cup C_4^{(3)})$. As every region of the considered subdrawing has at most two of the vertices a_2, b_2, c_2, and d_2 on its boundary, in D, at least two edges of $P^{(2)}$ joining $C_4^{(1)}$ with the vertices a_2, b_2, c_2, and d_2 cross the edges of $C_4^{(0)} \cup H^{(1)} \cup P^{(3)} \cup C_4^{(3)}$. But, the restriction $(\alpha,\beta) = 0$ does not allow a crossing between red edges of $C_4^{(0)} \cup P^{(1)}$ and blue edges of $P^{(2)}$, and the restriction $\beta^e = 1$ does not allow two crossings between blue edges and the edges of $P^{(3)} \cup C_4^{(3)}$. This confirms that there is no good drawing of the graph $P_n^2 \square C_4$ with less than $4(n-1)$ crossings. This completes the proof. □

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