MODELING HYPERCOMPLEX NUMBERS

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ABSTRACT
The present paper offers a generalization of the modeling by matrices for the complex numbers and quaternions to hypercomplex numbers of dimensions 2, 4 and 8. The given matrix model seems to be a suitable tool to study further properties of the hypercomplex numbers too. The matrix model used here was well known for modeling 2 dimensional complex and hypercomplex numbers, even for quaternions (see [4]), and we extend its use to the case of 4 and 8 dimensional hypercomplex numbers.

Keywords: *hypercomplex numbers, matrix modeling, non-commutative rings, Cayley-Dickson doubling*

1. INTRODUCTION

The operations of complex numbers are usually introduced by using their representation as points of the plane, and identifying the complex number $a+bi$ with the point $(a,b) \in \mathbb{R}^2$

$$(a,b)+(c,d) = (a+c,b+d)$$

and

$$(a,b)(c,d) = (ac-bd,ad+bc)$$

Another possibility to introduce this operations is to use the 2×2 real matrices. Really if we associate to $a+bi$ by a function

$$m:\mathbb{C}\to \boldsymbol{M}_2, \quad m(a+bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

$$\boldsymbol{M}_2 = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \middle| a,b\in\mathbb{R} \right\}$$

then the usual matrix operations will reflect exactly the properties of operations with complex numbers, in other words we established an isomorphism between the two structures: $(\mathbb{C},+,\cdot)$ and $(\boldsymbol{M}_2,+,\cdot)$ [4].

1.1. Ways to Higher Dimensions

What about higher dimensions? The Frobenius theorem states that the dimensions of division algebras are 1, 2, 4, and 8, if associativity for multiplication is required, then 1, 2, and 4, thus the possible algebras are isomorphic to the real numbers, to the complex numbers, the quaternions, and the Cayley numbers. The latter two are not commutative structures, and the last one is even not an associative one (it satisfies a weaker property – alternativity, i.e., $a\cdot(b\cdot b) = (a\cdot b)\cdot b$, and $(a\cdot a)\cdot b = a\cdot(a\cdot b)$). Another formulation is based on the properties related to the notion of conjugate, norm and absolute value which can be defined in all four structures. In $\mathbb{C}$ we will have:

$$|z_1z_2|^2 = |z_1|^2|z_2|^2, \quad z_j = a_j+b_ji, \quad j=1,2,$$

in coordinates

$$\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right) = (a_1a_2-b_1b_2)^2+(a_1b_2+a_2b_1)^2.$$

In a general context: $\sum\alpha_k^2\sum\beta_k^2 = \sum\gamma_k^2$, the product of the sum of n squares by the sum of other n squares is a sum of n squares.

Hurvitz's theorem afirms that the similar identity is possible only $n = 1, 2, 4, 8$.

W. Hamilton introduced in 1843 [2] two operations for the points of a four dimensional space $OXYZK$ such as their the induced operations on the coordinate planes XOY, XOZ, and XOK coincide with the operations of the complex numbers. This was the first example of non-commutative field and had an impact on the developments of mathematics and physics.

This structure nowadays is called **Hamilton-quaternions skew field** denoted by $\mathbb{H}$, and its elements are the quaternions.

Let be 1, i, j, k a base for $\mathbb{H}$, and consider in $\mathbb{H}$ the following identities for the base elements, as usual:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k,$$

$$jk = -kj = i, \quad ki = -ik = j.$$

The product of two arbitrary elements of $\mathbb{H}$ can be defined as follows:

$$(a+bi+cj+dk)(e+fi+gj+hk) =$$

$$(ae-bf-cg-dh)+(af+be+ch-dg)i+$$
$$+(ag+ce-bh+df)j+(ah+de+bg-cf)k.$$

The structure of $\mathbb{H}$ will be an associative, with unit $1+0i+0j+0k$ which will be shortly denoted by 1. Based on the definition of the multiplication it can be seen that for any quaternion $x = a+bi+cj+dk \neq 0$, it will exist a unique inverse

$$x^{-1} = \frac{a-bi-cj-dk}{a^2+b^2+c^2+d^2}.$$

If we denote by $\bar{x} = a-bi-cj-dk$ the conjugated quaternion of $x = a+bi+cj+dk$ we can see in an easy way that both $x+\bar{x}$ and $x\bar{x}$ are real, just as in the case of complex numbers. We can introduce the norm of a quaternion as $\|x\| = x\bar{x}$. Moreover

$$x^{-1} = \frac{\bar{x}}{x\bar{x}}$$

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Similar to the complex numbers, we can introduce the matrix model for quaternions:

$$a+bi+cj+dk \mapsto \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix}$$

will define a bijection $(\mathbb{H},+,\cdot) \to (\boldsymbol{M}_4,+,\cdot)$, where

$$\boldsymbol{M}_4 = \left\{ \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{bmatrix} \middle| a,b,c,d \in \mathbb{R} \right\}.$$

Remark 1.1. *This model sometimes appears as 2×2 complex matrix, based on the so called the Cayley-Dickson construction (doubling), in this case $x = a+bi+cj+dk$ is written as $u+vj$, where $u = a+bi, v = c+di$, $i^2 = j^2 = -1$, $k = ij$. Now we can take $u+vj \mapsto \begin{bmatrix} u & v \\ -\bar{v} & \bar{u} \end{bmatrix}$, as the matrix model of quaternions.*

2. HYPERCOMPLEX NUMBERS

Kantor and Solodovnikov [3] introduced hypercomplex numbers, as algebras over the real numbers in the following way:

$$\mathbb{H} = \Big\{ a_0 + a_1 i_1 + a_2 i_2 + \ldots + a_n i_n \mid a_i \in \mathbb{R}, \\ i_k^2 \in \{-1,0,1\} \Big\}$$

where a_0, a_1, a_2, ..., a_n are real coefficients, and $\{1, i_1, i_2, \ldots, i_n\}$ is a normalized base, i.e., $i_k^2 \in \{-1,0,1\}$.

For $n = 1$ there are 3 possibilities.

$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}, i^2 = -1\}$ the complex numbers.

$\mathbb{S} = \{a+bE \mid a,b \in \mathbb{R}, E^2 = 1\}$ the hyperbolic complex numbers (split-complex).

$\mathbb{D} = \{a+b\Omega \mid a,b \in \mathbb{R}, \Omega^2 = 0\}$ the dual-complex numbers, or Study-numbers.

Of course only one, the complex numbers is a commutative field, the other two are commutative rings only.

These numbers have many important applications, like in the description of the Lorentz transformations (see, e.g., [1]).

This rings can be modeled in a similar way to the complex numbers with 2×2 real matrices in the following way:

In $\mathbb{S}$:

$$m(a+bE) = \begin{bmatrix} a & b \\ b & a \end{bmatrix},\ m(E) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

and

$$m(E^2) = (m(E))^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

that is $1+0j = \mathbf{1}$.

In $\mathbb{D}$:

$$m(a+b\Omega) = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix},\ m(\Omega) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

and

$$m(\Omega^2) = (m(\Omega))^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

that is $0+0\Omega = \mathbf{0}$.

These models help us to see that $\mathbb{S}$ and $\mathbb{D}$ have the structure of commutative rings with unit indeed:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} c & d \\ d & c \end{bmatrix} = \begin{bmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{bmatrix},$$

$$\begin{bmatrix} c & d \\ d & c \end{bmatrix}\begin{bmatrix} a & b \\ b & a \end{bmatrix} = \begin{bmatrix} ac+bd & ad+bc \\ ad+bc & ac+bd \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} c & d \\ 0 & c \end{bmatrix} = \begin{bmatrix} ac & ad+bc \\ 0 & ac \end{bmatrix},$$

$$\begin{bmatrix} c & d \\ 0 & c \end{bmatrix}\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} ac & ad+bc \\ 0 & ac \end{bmatrix}.$$

The properties of the conjugated and the absolute value are “inherited” as well,

$$\overline{a+bE} = a-bE, \quad \text{and} \quad \overline{a+b\Omega} = a-b\Omega,$$

and we have:

$$a+bE+\overline{a+bE} = 2a \in \mathbb{R},$$

and

$$a+b\Omega+\overline{a+b\Omega} = 2a \in \mathbb{R},$$

moreover:

$$(a+bE)\left(\overline{a+bE}\right) = a^2-b^2 \in \mathbb{R},$$

and

$$(a+b\Omega)\left(\overline{a+b\Omega}\right) = a^2 \in \mathbb{R},$$

because:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}\begin{bmatrix} a & -b \\ -b & a \end{bmatrix} = \begin{bmatrix} a^2-b^2 & 0 \\ 0 & a^2-b^2 \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}\begin{bmatrix} a & -b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix}.$$

The first problem we observe is the existence of the inverse, which do not exist for all such numbers. For hyperbolic complex numbers the inverse will be defined only if $a^2-b^2 \neq 0$:

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix}^{-1} = \begin{bmatrix} \frac{a}{a^2-b^2} & -\frac{b}{a^2-b^2} \\ -\frac{b}{a^2-b^2} & \frac{a}{a^2-b^2} \end{bmatrix},$$

and for Study-numbers $a+b\Omega$ we have a similar condition $a^2 \neq 0$:

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{a} & -\frac{b}{a^2} \\ 0 & \frac{1}{a} \end{bmatrix} = \begin{bmatrix} \frac{a}{a^2} & -\frac{b}{a^2} \\ 0 & \frac{a}{a^2} \end{bmatrix}.$$

3. DOUBLING THE HYPERCOMPLEX NUMBERS

Using the Cayley-Dickson construction, we can study the numbers of the form:

$$x = a + bX + (c + dX)Y, \quad (a,b,c,d \in \mathbb{R}),$$

where

$$X^2 \in \{-1,0,1\}, \quad Y^2 \in \{-1,0,1\}, \quad \text{and} \quad Z = XY.$$

What about the value of Z^2 in the different cases?

Case 1. Hyperbolic-complex numbers

$$x = a + bE + (c + dE)F = a + bE + cF + dG,$$
$$(a,b,c,d \in \mathbb{R}),$$
$$E^2 = 1, F^2 = 1, G^2 = -1, EF = -FE = G,$$
$$EG = -GE = F, GF = -FG = E,$$

form a non-commutative ring with unit, and the above identities are satisfied.

The model to be used is

$$m(a + bE + cF + dG) = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ c & -d & a & -b \\ -d & c & -b & a \end{bmatrix},$$

and if we have the units:

$$1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$$F = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

we can compute

$$E^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1,$$

similarly

$$F^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 1,$$

however $G^2 = -1$:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = -1.$$

Let us check the other products too.

$EF = G$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix},$$

$FE = -G$:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$GF = E$:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix},$$

$FG = -E$:

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$EG = F$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$GE = -F$:

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

What about conjugated numbers and inverses?

$$A + \overline{A} = 2a, \quad A\overline{A} = a^2 - b^2 - c^2 + d^2,$$

$$A^{-1} = \frac{1}{a^2 - b^2 - c^2 + d^2} \begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & -a & b \\ d & -c & b & a \end{bmatrix},$$

this exists only if $a^2 - b^2 - c^2 + d^2 \neq 0$.

Case 2. Dual-complex numbers

In a similar way we can double the dual-complex numbers in the form

$$x = a + b\Omega + (c + d\Omega)\Phi, \quad (a,b,c,d \in \mathbb{R}),$$

if we put $\Omega^2 = 0$, $\Phi^2 = 0$, and $\Theta = \Omega\Phi$.

What about the value of Θ^2 now? We will use similarly:

$$m(a + b\Omega + c\Phi + d\Theta) = \begin{bmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & -b \\ 0 & 0 & 0 & a \end{bmatrix}$$

and obtain, that the numbers

$$x = a + b\Omega + (c + d\Omega)\Phi =$$
$$a + b\Omega + c\Phi + d\Theta, \quad (a,b,c,d \in \mathbb{R}),$$

where $\Omega^2 = 0$, $\Phi^2 = 0$, and $\Theta^2 = 0$, $\Omega\Phi = -\Phi\Omega = \Theta$, $\Phi\Theta = \Theta\Phi = \Omega\Theta = \Theta\Omega = \mathbf{0}$ form a non-commutative ring as well.

Case 3. Dual-form complex numbers

The dual-form complex numbers are doubling the complex numbers, but in the dual-complex form,

$$x = a + bX + cY + dZ, \quad (a,b,c,d \in \mathbb{R}),$$

where we put $X^2 = -1$, $Y^2 = 0$, $Z^2 = 0$, $XY = -YX = Z$, $YZ = -ZY = 0$, $ZX = -XZ = 0$, and they form a noncommutative ring with unit, too. The model to be used here is

$$m(a + bX + cY + dZ) = \begin{bmatrix} a & b & c & d \\ -b & a & -d & c \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}.$$

Case 4. Dual-form hyperbolic complex numbers

The numbers of the form

$$x = a + bU + cV + dW, \quad (a,b,c,d \in \mathbb{R}),$$

satisfying the identities $U^2 = 1$, $V^2 = 0$, $W^2 = 0$, $UV = -VU = W$, $VW = -WV = 0$, $UW = -WU = V$ form a noncommutative ring with unit. The model to be used here is

$$m(a + bU + cV + dW) = A = \begin{bmatrix} a & b & c & d \\ b & a & d & c \\ 0 & 0 & a & -b \\ 0 & 0 & -b & a \end{bmatrix}.$$

Remark 3.1. ***The Cayley numbers***
Using the Cayley-Dickson construction for quaternions we can define the Cayley numbers, but this time even the associativity is to be given up, the Cayley numbers form a so called alternative ring, that is for the multiplication is not anymore true the usual associativity, but a weaker property:

$$(u \cdot v) \cdot w \neq u \cdot (v \cdot w),$$

but

$$(u \cdot v) \cdot v = u \cdot (v \cdot v), \quad \textit{and} \quad v \cdot (v \cdot u) = (v \cdot v) \cdot u.$$

So if we introduce the doubling of quaternions

$$x = (a + bi + cj + dk) + (e + fi + gj + hk)\Gamma$$

we will have numbers of the form

$$x = a + b \cdot i + c \cdot j + d \cdot k + e \cdot E + f \cdot I + g \cdot J + h \cdot K,$$

the Cayley numbers and the respective identities now can be introduced in an 8×8 *table.*

In the case of Cayley numbers, it can be introduced the conjugate, the norm, the inverse, but the multiplication is neither commutative nor associative.

The matrix models used in the previous cases cannot be extended for Cayley numbers, as the structure of matrices is associative.

Case 5. Dual quaternions

If we remind the four dimensional cases it can be formulated the idea to use a doubling by "mixture", that is doubling the quaternions like dual-complex numbers. For this we may consider a 2×2 quaternion matrix like:

$$\begin{bmatrix} u & v \\ 0 & \overline{u} \end{bmatrix},$$

where u and v are quaternions, or an 8×8 matrix as follows:

$$A = \begin{bmatrix} a & b & c & d & e & f & g & h \\ -b & a & -d & c & -f & e & -h & g \\ -c & d & a & -b & -g & h & e & -f \\ -d & -c & b & a & -h & -g & f & e \\ 0 & 0 & 0 & 0 & a & -b & -c & -d \\ 0 & 0 & 0 & 0 & b & a & d & -c \\ 0 & 0 & 0 & 0 & c & -d & a & b \\ 0 & 0 & 0 & 0 & d & c & -b & a \end{bmatrix},$$

and $\det A = \left(a^2 + b^2 + c^2 + d^2\right)^4 \neq 0$.

This is an associative, noncommutative ring, and the inverse will exist if the "quaternion part" is different from 0.

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