THREE-SEGMENT PIECEWISE-LINEAR VECTOR FIELDS WITH ORTHOGONAL EIGENSPACES

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ABSTRACT

In the upcoming article eight new members of the dynamical systems of class C and F are investigated via numerical integration. Although four configurations of the state matrices are familiar with the chaos generation the others are hypothetic and complete the study. The aim of this paper is to discover and describe different canonical representations for a given set of the differential equations, i.e. existing mathematical model. Individual dynamical systems are distinct from the viewpoint of the geometry of the associated vector field.

The motivation for discovering the new mathematical models capable to produce a complex dynamics including chaos is both pedagogical and practical. The latter case is obvious since the new system can be easier to be implemented as an electronic circuit or has other advantage like simpler location of strategic orbits.

Keywords: *dynamical system, eigenvalues, geometry, vector field, chaos.*

1. INTRODUCTION

It is well known that many real physical systems evolves with time and thus can be described by a set of the differential equations. Assume the extensive group of the autonomous deterministic dynamical systems given by a compact matrix equation [2]

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{b}\,\mathrm{h}\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right), \tag{1}$$

where the scalar saturation-type nonlinear function

$$\mathrm{h}\left(\mathbf{w}^{\mathrm{T}}\mathbf{x}\right) = \frac{1}{2}\left(\left|\mathbf{w}^{\mathrm{T}}\mathbf{x}+1\right|-\left|\mathbf{w}^{\mathrm{T}}\mathbf{x}-1\right|\right), \tag{2}$$

separates the state space by two parallel boundary planes $\mathbf{w}^{\mathrm{T}}\mathbf{x}=\pm 1$ into the three affine regions

$$\left|\mathbf{w}^{\mathrm{T}}\mathbf{x}\right| > 1 \Rightarrow \dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \pm \mathbf{b},\quad \left|\mathbf{w}^{\mathrm{T}}\mathbf{x}\right| < 1 \Rightarrow \dot{\mathbf{x}} = \left(\mathbf{A}+\mathbf{b}\mathbf{w}^{\mathrm{T}}\right)\mathbf{x}\,. \tag{3}$$

Since the dynamical motion in both outer segments is described by the same characteristic polynomial the corresponding eigenvalues as well as geometry of the vector field is identical.

The three-segment piecewise-linear vector field is a typical feature of the so-called Chua´s oscillator deeply analysed in the publication [1]. Thanks to this prototype dynamical system many other members of the same group can be derived using the concept called linear topological conjugacy. Two simpler forms compared to the original mathematical model have been published in [2]. Having this basic group of the dynamical systems, the reference model has been introduced in [3] and also numerically verified. The complex decomposed and elementary canonical models are briefly discussed in [4]. The essential problem to obtain other dynamical systems with qualitatively same dynamics can be solved by using a Cayley-Hamilton´s theorem, namely the fact that every square matrix of the real numbers must annulate its own characteristic polynomial.

This theorem can be writteln in the form

$$\Phi(\lambda) = \det\left(\lambda\mathbf{E} - \mathbf{A}\right) \;\Rightarrow\; \Phi(\mathbf{A}) = \mathbf{0}\,. \tag{4}$$

In this equation $\mathbf{E}$ is the unity matrix and $\mathbf{0}$ is a zero matrix. Let suppose one dynamical system described by an expression (1) together with (2) to be marked as $\mathbf{A}_1$, $\mathbf{b}_1$, $\mathbf{w}_1$ and the second dynamical system belonging to the same class as $\mathbf{A}_2$, $\mathbf{b}_2$, $\mathbf{w}_2$. The partial transformation matrix denoted as $\mathbf{K}_i$ transforms arbitrary i-th dynamical system into its normal form

$$\mathbf{K}_1\mathbf{A}_1\mathbf{K}_1^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta_3 & -\eta_2 & \eta_1 \end{pmatrix},\quad \mathbf{K}_2\mathbf{A}_2\mathbf{K}_2^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \eta_3 & -\eta_2 & \eta_1 \end{pmatrix}, \tag{5}$$

where $\eta_{1,2,3}$ are coefficients of the characteristic polynomial. It is evident that if the global behavior of the two systems is required to be the same, these real-valued numbers must be also exactly the same for both systems. The entire transformation process $\mathbf{T}$ can be acquired by a composition of the two particular transforms $\mathbf{K}_1$ and $\mathbf{K}_2$, in detail

$$\mathbf{T} = \mathbf{K}_1^{-1}\mathbf{K}_2 = \begin{pmatrix} \mathbf{w}_1^{\mathrm{T}} \\ \mathbf{w}_1^{\mathrm{T}}\mathbf{A}_1 \\ \mathbf{w}_1^{\mathrm{T}}\mathbf{A}_1^2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{w}_2^{\mathrm{T}} \\ \mathbf{w}_2^{\mathrm{T}}\mathbf{A}_2 \\ \mathbf{w}_2^{\mathrm{T}}\mathbf{A}_2^2 \end{pmatrix}. \tag{6}$$

This is the way how to obtain complete system if the state matrix $\mathbf{A}$ is already defined. Let suppose we are looking for some specific form of the state matrix in inner segment of the vector field. The final system can be generally expressed by means of the first one

$$\dot{\tilde{\mathbf{x}}} = \mathbf{T}^{-1}\mathbf{A}\,\mathbf{T}\,\tilde{\mathbf{x}} + \mathbf{T}^{-1}\mathbf{b}\,\mathrm{h}\left(\mathbf{w}^{\mathrm{T}}\mathbf{T}\,\tilde{\mathbf{x}}\right). \tag{7}$$

Equivalently if the state matrix $\mathbf{A}$ is given then $\mathbf{K}$ should be computed. The matrix $\mathbf{K}_1$ of the reference model equals to the unity matrix thus $\mathbf{T}=\mathbf{K}_2$. The first canonical equivalent [2] is also well suited as starting system due to the very simple form of $\mathbf{K}_1$.

2. INDIVIDUAL RESULTS

In this chapter, particular dynamical systems will be introduced by a symbolical expressions with the eigenvalues as parameters, namely $\mu'\pm\mu''j$ are com-plex conjugated eigenvalues in the inner segment of the vector field and $\mu_{1,2,3}$ are real eigenvalues in the same region. Similarly, $\upsilon'\pm\upsilon''j$ are complex conju-gated eigenvalues in the outer segment of the vector field and $\upsilon_{1,2,3}$ are real eigenvalues at the same place. We should also adopt the two-letter notion for each derived system. First letter denotes the geometry of the vector field in the outer segment and second one describes the inner segment. To be more specific, if the letter C is used then there exists a combination of eigenplane and eigenvector and letter F suggests three distinct eigenvectors. To this end, underlined letter means that the associated invariant manifolds are orthogonal.

The numerical integration of individual systems of differential equations was done by using Mathcad and build-in fourth-order Runge-Kutta method with initial conditions $\mathbf{x}_0=(0.1\ 0\ 0)^T$, step size $\Delta t=0.1$ and fixed final time $t_{max}=1000$. Several typical chaotic attractors are mentioned below. The dynamical behavior of individual systems are given exclusively by the set of eigenvalues. In detail, the double-scroll attractor (DSA) is generated for

$$\upsilon'=0.061 \quad \upsilon''=1 \quad \upsilon_3=-1.29 \quad \mu'=-0.319 \quad \mu''=0.892 \quad \mu_3=0.728$$

The dual double-scroll attractor (DDSA) has the opposite stability indexes of the equilibria in each segment of the vector field, i.e. 1→2 and 2→1

$$\upsilon'=-0.987 \quad \upsilon''=2.59 \quad \upsilon_3=2.26 \quad \mu'=0.206 \quad \mu''=3 \quad \mu_3=-3.984$$

The last dynamical system with a pair of complex conjugated eigenvalues in each region of the state space will be called trumpet attractor (TA). Note that the equilibria in the inner segment of the vector field is truly unstable. It means that the trajectory in the inner segment is repelled along each direction of the eigenspace

$$\upsilon'=0.3 \quad \upsilon''=1 \quad \upsilon_3=-3 \quad \mu'=0.3 \quad \mu''=10 \quad \mu_3=0.2$$

For the real eigenvalues in the inner segment of the vector field the double-hook attractor (DHA) exists for the following set

$$\upsilon'=0.18 \quad \upsilon''=1.15 \quad \upsilon_3=-1.04 \quad \mu_1=-5.6 \quad \mu_2=-3.4 \quad \mu_3=1.25$$

Assume the same geometry of the vector field with apparent changes in stability indexes. The double-funnel attractor (DFA) can be observed for

$$\upsilon'=-0.2 \quad \upsilon''=1 \quad \upsilon_3=0.02 \quad \mu_1=1.032 \quad \mu_2=0.135 \quad \mu_3=-0.443$$

In further text, particular attractors will be called using its letter shortcuts.

2.1. System $\underline{\text{C}}$C

Roughly speaking we are looking for the specific matrix expressions of the dynamical systems in the individual chapters of this paper. In accordance with the publication [4] choosing the vector $\mathbf{w}=(1\ 0\ 1)^T$ the partial and complete transformation matrix has the form

$$\mathbf{K}_2=\begin{pmatrix}1 & 0 & 1\\ \upsilon' & -\upsilon'' & \upsilon_3\\ \upsilon'^2-\upsilon''^2 & -2\upsilon'\upsilon'' & \upsilon_3^2\end{pmatrix},\ \mathbf{T}=\begin{pmatrix}1 & 0 & 1\\ \upsilon'+\upsilon_3 & \upsilon'' & 2\upsilon'\\ \upsilon'\upsilon_3 & \upsilon''\upsilon_3 & \upsilon'^2+\upsilon''^2\end{pmatrix}. \quad (8)$$

This form of the transformation results into the state space equations with matrix $\mathbf{A}$ in the Jordan form

$$\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}=\begin{pmatrix}\upsilon' & -\upsilon'' & 0\\ \upsilon'' & \upsilon' & 0\\ 0 & 0 & \upsilon_3\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}+\begin{pmatrix}\tilde{b}_1\\ \tilde{b}_2\\ \tilde{b}_3\end{pmatrix}h\left(\begin{pmatrix}1 & 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right), \quad (9)$$

where individual elements of the column vector $\mathbf{T}^{-1}\mathbf{b}$ can be expressed as

$$\begin{aligned}
\tilde{b}_1 &= \left(\upsilon'^2+\upsilon''^2-2\upsilon'\upsilon_3\right)\left(2\mu'+\mu_3-2\upsilon'-\upsilon_3\right)/\xi+\\
&+\upsilon_3\left(\mu'^2+\mu''^2-2\mu'\mu_3-\upsilon'^2+\upsilon''^2-2\upsilon'\upsilon_3\right)/\xi+\\
&-\mu_3\left(\mu'^2+\mu''\right)+\upsilon_3\left(\upsilon'^2+\upsilon''^2\right)/\xi\\
\tilde{b}_2 &= -\left(\upsilon'^3+\upsilon'\upsilon''^2-\upsilon'^2\upsilon_3+\upsilon''^2\upsilon_3\right)\left(2\mu'+\mu_3-2\upsilon'-\upsilon_3\right)/\left(\upsilon''\xi\right)+\\
&+\left(\upsilon'^2+\upsilon''^2-\upsilon'\upsilon_3\right)\left(\mu'^2+\mu''^2+2\mu'\mu_3-\upsilon'^2+\upsilon''^2-2\upsilon'\upsilon_3\right)/\upsilon''\xi-\\
&-\left(\upsilon'-\upsilon_3\right)\left[\mu_3\left(\mu'^2+\mu''^2\right)-\upsilon_3\left(\upsilon'^2+\upsilon''^2\right)\right]/\left(\upsilon''\xi\right)\\
\tilde{b}_3 &= \left[\upsilon_3^2\left(2\mu'+\mu_3-2\upsilon'-\upsilon_3\right)+\mu_3\left(\mu'^2+\mu''^2\right)\right]/\xi-\\
&-\upsilon_3\left(\mu'^2+\mu''^2+2\mu'\mu_3-2\upsilon'\upsilon_3\right)/\xi
\end{aligned} \quad (10)$$

where the following denominator constant has been introduced

$$\xi=\left(\upsilon_3-\upsilon'\right)^2+\upsilon''^2. \quad (11)$$

In further text a vector $\mathbf{w}=(1\ 1\ 1)^T$ has been chosen. It can be prooved that this is also a correct choice leading to the different form for this class of dyna-mical system, in detail

$$\mathbf{K}_2=\begin{pmatrix}1 & 1 & 1\\ \upsilon'+\upsilon'' & \upsilon'-\upsilon'' & \upsilon_3\\ \upsilon'^2-\upsilon''^2+2\upsilon'\upsilon'' & \upsilon'^2-\upsilon''^2-2\upsilon'\upsilon'' & \upsilon_3^2\end{pmatrix},$$

$$\mathbf{T}=\begin{pmatrix}1 & 1 & 1\\ \upsilon'+\upsilon_3-\upsilon'' & \upsilon'+\upsilon_3+\upsilon'' & 2\upsilon'\\ \upsilon_3\left(\upsilon'-\upsilon''\right) & \upsilon_3\left(\upsilon'+\upsilon''\right) & \upsilon'^2+\upsilon''^2\end{pmatrix}, \quad (12)$$

$$\mathbf{T}^{-1}=\begin{pmatrix}\dfrac{\xi_1}{2\upsilon''\xi_3} & \dfrac{\upsilon_3\left(\upsilon'+\upsilon''\right)-\upsilon'^2-\upsilon''^2}{2\upsilon''\xi_3} & \dfrac{\upsilon'-\upsilon''-\upsilon_3}{2\upsilon''\xi_3}\\ \dfrac{\xi_2}{2\upsilon''\xi_3} & \dfrac{\upsilon_3\left(\upsilon''-\upsilon'\right)+\upsilon'^2+\upsilon''^2}{2\upsilon''\xi_3} & \dfrac{\upsilon'+\upsilon''-\upsilon_3}{2\upsilon''\xi_3}\\ \upsilon_3^2/\xi_3 & -\upsilon_3/\xi_3 & 1/\xi_3\end{pmatrix}$$

where auxiliary constants are

$$\xi_1 = \upsilon'^3 + \upsilon''^3 - \upsilon_3\left(\upsilon'^2 - \upsilon''^2\right) + \upsilon''\left(\upsilon'^2 - 2\upsilon'\upsilon_3\right) + \upsilon'\upsilon'' ,$$

$$\xi_2 = \upsilon''^3 - \upsilon'^3 + \upsilon_3\left(\upsilon'^2 - \upsilon''^2\right) + \upsilon''\left(\upsilon'^2 - 2\upsilon'\upsilon_3\right) - \upsilon'\upsilon'' , \qquad (13)$$

$$\xi_3 = \upsilon'^2 + \upsilon''^2 - 2\upsilon'\upsilon_3 + \upsilon_3^2 .$$

After making an operation $\mathbf{T}^{-1}\mathbf{b}$ we get immediately

$$\tilde{b}_1 = \left[-\upsilon'^4 + \upsilon'^3\left(\upsilon_3 + 2\mu' + \mu_3 - 2\upsilon''\right)\right]/\xi$$
$$+\upsilon'^2\left(\upsilon''(3\upsilon_3 + \mu_3 + 2\mu') - \mu_3(2\mu' + \upsilon_3) - 2\upsilon_3\mu' - \mu'^2 - \mu''^2\right)/\xi +$$
$$+\upsilon'\left[\upsilon''^2(2\mu' - 3\upsilon_3 + \mu_3) - \upsilon''(4\upsilon_3\mu' + 2\upsilon_3\mu_3) - 2\upsilon''^3\right]/\xi +$$
$$+\upsilon'\left[\upsilon_3\left(\mu'^2 + \mu''^2 + 2\mu'\mu_3\right) + \mu_3\left(\mu'^2 + \mu''^2\right)\right]/\xi +$$
$$+\left[\upsilon''^4 + \upsilon''^3(\mu_3 - \upsilon_3 + 2\mu') + \upsilon''\left(\upsilon_3\left(\mu'^2 + \mu''^2 + 2\mu'\mu_3\right)\right)\right]/\xi +$$
$$+\upsilon''^2\left(\mu_3(\upsilon_3 - 2\mu') + 2\upsilon_3\mu' - \mu'^2 - \mu''^2\right)/\xi -$$
$$-\left[\upsilon''\left(\mu_3\left(\mu'^2 + \mu''^2\right)\right) + \upsilon_3\mu_3\left(\mu'^2 + \mu''^2\right)\right]/\xi$$

$$\tilde{b}_2 = \left[\upsilon'^4 - \upsilon'^3\left(\upsilon_3 + 2\mu' + \mu_3 + 2\upsilon''\right)\right]/\xi +$$
$$\tilde{b}_2 = +\upsilon'^2\left(\upsilon''(3\upsilon_3 + \mu_3 + 2\mu') + \mu_3(2\mu' + \upsilon_3) + 2\upsilon_3\mu' + \mu'^2 + \mu''^2\right)/\xi +$$
$$+\upsilon'\left[\upsilon''^2(3\upsilon_3 - 2\mu' - \mu_3) - \upsilon''(4\upsilon_3\mu' + 2\upsilon_3\mu_3)\right]/\xi -$$
$$-\upsilon'\left[2\upsilon''^3 + \upsilon_3\left(\mu'^2 + \mu''^2 + 2\mu'\mu_3\right) + \mu_3\left(\mu'^2 + \mu''^2\right)\right]/\xi +$$
$$+\left[-\upsilon''^4 + \upsilon''^3(\mu_3 - \upsilon_3 + 2\mu') + \upsilon''\left(\upsilon_3\left(\mu'^2 + \mu''^2 + 2\mu'\mu_3\right)\right)\right]/\xi +$$
$$+\upsilon''^2\left(\mu_3(2\mu' - \upsilon_3) - 2\upsilon_3\mu' + \mu'^2 + \mu''^2\right)/\xi -$$
$$-\left[\upsilon''\left(\mu_3\left(\mu'^2 + \mu''^2\right)\right) - \upsilon_3\mu_3\left(\mu'^2 + \mu''^2\right)\right]/\xi$$

$$\tilde{b}_3 = \frac{\left(\mu'^2 + \mu''^2\right)(\mu_3 - \upsilon_3) + \upsilon_3^2(2\mu' + \mu_3 - \upsilon_3) - 2\upsilon_3\mu'\mu_3}{\xi}$$

Similarly to the previous derivation process the auxiliary constant is introduced in order to simplify the expression. The mentined constant is

$$\xi = 2\upsilon''\left[(\upsilon' - \upsilon_3)^2 + \upsilon''^2\right] . \qquad (15)$$

Remember that although the symbolical expression of the new dynamical system can (and often is) quite complicated its numerical equivalent is taken for the practical applications. For example, if an engineer decides to implement (9) together with (14) as an electronic circuit he must choose the desired state space attractor, adopt the corresponding eigenvalues and numerically calculate each element of **A** and **b** in the matrix equation (1). Clearly some configura-tions of the eigenvalues must be excluded from the computation since dividing by zero can occur in the complicated terms (14).

For the second variant of the dynamical system DSA, DDSA and TA is visible in Fig. 1, Fig. 2 and Fig. 3 respectively. These are the three-dimensional perspective plots of the state space trajectory.

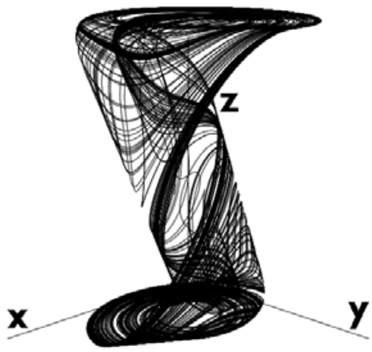

Fig. 1 DSA generated by C$\underline{\text{C}}$ dynamical system.

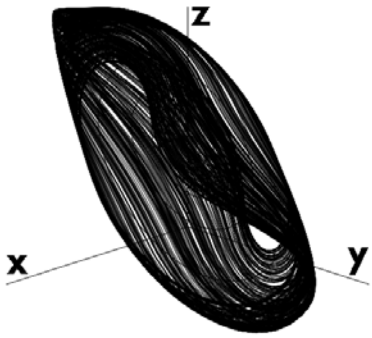

Fig. 2 DDSA generated by C$\underline{\text{C}}$ dynamical system.

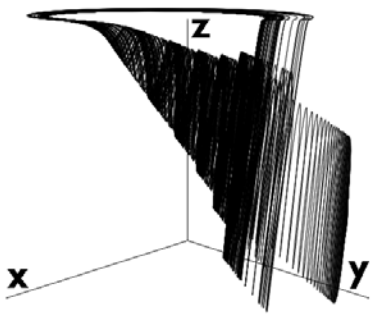

Fig. 3 TA generated by C$\underline{\text{C}}$ dynamical system.

2.2. System C$\underline{\text{C}}$

Starting with the given state matrix $\mathbf{A}+\mathbf{b}\mathbf{w}^T$ in the inner segment of the vector field the complete trans-formation into the Jordan form can be derived. Thus the individual columns of the matrix **T** are directly eigenvectors. For example, transformation matrix **T** from the elementally canonical and block diagonal model given in [3] and its inversion is

$$\mathbf{T} = \begin{pmatrix} \dfrac{\mu' - \mu_3}{\xi_1} & \dfrac{\mu''}{\xi_1} & 0 \\ \dfrac{\mu'(\mu' - \mu_3) + \mu''^2}{\xi_1} & \dfrac{\mu''\mu_3}{\xi_1} & 0 \\ 1/\sqrt{\dfrac{\xi_2}{(\mu_3 - \upsilon_3)^2}} & 0 & 1 \end{pmatrix} , \qquad (16)$$

$$\mathbf{T}^{-1} = \frac{\xi_2}{\xi_3}\begin{pmatrix} \frac{\mu''\mu_3}{\xi_1} & \frac{-\mu''}{\xi_1} & 0 \\ \frac{\mu'\mu_3-\mu'^2-\mu''^2}{\xi_1} & \frac{\mu'-\mu_3}{\xi_1} & 0 \\ \mu''\mu_3\frac{\upsilon_3-\mu_3}{\xi_2} & \mu''\frac{\mu_3-\upsilon_3}{\xi_2} & \mu''\frac{\xi_3}{\xi_2} \end{pmatrix},$$

where ξ_1, ξ_2 and ξ_3 is given as

$$\xi_1 = \left(\mu_3-\upsilon_3\right)\sqrt{\frac{\xi_2}{\left(\mu_3-\upsilon_3\right)^2}}\,,$$

$$\xi_2 = \left(\mu'-\mu_3\right)^2\left(\mu_3-\upsilon_3\right)^2 + \left(\mu'^2+\mu''^2\right)^2 +$$

$$+\,\mu'^2\mu_3\left(\mu_3-2\mu'\right) + \mu''^2\left(\mu_3^2+1-2\mu'\mu_3\right),$$

$$\xi_3 = \mu''\left(2\mu'\mu_3-\mu'^2-\mu''^2-\mu_3\right). \qquad (17)$$

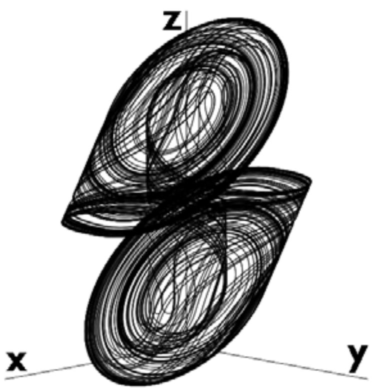

Fig. 4 DSA generated by C<u>C</u> dynamical system.

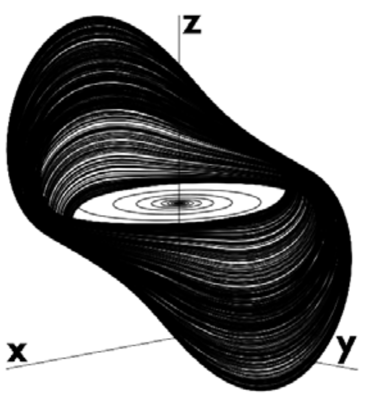

Fig. 5 DDSA generated by C<u>C</u> dynamical system.

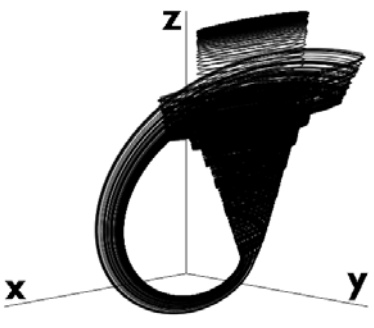

Fig. 6 TA generated by C<u>C</u> dynamical system.

The state equations of new dynamical system can be obtained by symbolical examination of $\mathbf{T}^{-1}\mathbf{AT}$, $\mathbf{T}^{-1}\mathbf{b}$ and $\mathbf{Tw}$ in the equation (7) with

$$\mathbf{A} = \begin{pmatrix} 2\upsilon' & -1 & 2\left(\upsilon'-\mu'\right) \\ \upsilon'^2+\upsilon''^2 & 0 & \upsilon'^2+\upsilon''^2-\mu'^2-\mu''^2 \\ 0 & 0 & \upsilon_3 \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} 2\left(\mu'-\upsilon'\right) \\ \mu'^2+\mu''^2-\upsilon'^2-\upsilon''^2 \\ \mu_3-\upsilon_3 \end{pmatrix}, \quad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \qquad (18)$$

Such operation leads to very complicated formulas. DSA, DDSA as well as TA generated by this system is shown in Fig. 4, Fig. 5 and Fig. 6 respectively.

2.3. System <u>F</u>C

This configuration of the state matrices repre-sents first hypothetic case of the dynamical systems under inspection. The corresponding state equations in the compact matrix form are

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \upsilon_1 & 0 & 0 \\ 0 & \upsilon_2 & 0 \\ 0 & 0 & \upsilon_3 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \tilde{b}_3 \end{pmatrix} h\left(\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right). \qquad (19)$$

It is not hard to learn that starting with the same vector **w** as in the case of system (9) leads to the singular matrix **K**. Choosing the vector $\mathbf{w}=(1\ 1\ 1)^T$ the associated boundary planes are not parallel to any eigenspace giving a promising result

$$\mathbf{K}_2 = \begin{pmatrix} 1 & 1 & 1 \\ \upsilon_1 & \upsilon_2 & \upsilon_3 \\ \upsilon_1^2 & \upsilon_2^2 & \upsilon_3^2 \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 1 & 1 & 1 \\ \upsilon_2+\upsilon_3 & \upsilon_1+\upsilon_3 & \upsilon_1+\upsilon_2 \\ \upsilon_2\upsilon_3 & \upsilon_1\upsilon_3 & \upsilon_1\upsilon_2 \end{pmatrix}, \qquad (20)$$

$$\mathbf{T}^{-1} = \begin{pmatrix} \upsilon_1^2/\xi_1 & -\upsilon_1/\xi_1 & 1/\xi_1 \\ \upsilon_2^2/\xi_2 & -\upsilon_2/\xi_2 & 1/\xi_2 \\ \upsilon_3^2/\xi_3 & -\upsilon_3/\xi_3 & 1/\xi_3 \end{pmatrix},$$

where denominator constants are

$$\begin{aligned} \xi_1 &= \upsilon_1^2 + \upsilon_2\upsilon_3 - \upsilon_1\left(\upsilon_2+\upsilon_3\right), \\ \xi_2 &= \upsilon_2^2 + \upsilon_1\upsilon_3 - \upsilon_2\left(\upsilon_1+\upsilon_3\right), \\ \xi_3 &= \upsilon_3^2 + \upsilon_1\upsilon_2 - \upsilon_3\left(\upsilon_1+\upsilon_2\right). \end{aligned} \qquad (21)$$

Applying the transformation **T** and its inverse inside the equation (6) we get immediatelly

$$\begin{aligned} \tilde{b}_1 &= \frac{\upsilon_1^2\left(2\mu'+\mu_3\right)-\upsilon_1^3-2\,\upsilon_1\mu'\mu_3+\left(\mu_3-\upsilon_1\right)\left(\mu'^2+\mu''^2\right)}{\upsilon_2\upsilon_3-\upsilon_1\upsilon_3+\upsilon_1^2-\upsilon_1\upsilon_2} \\ \tilde{b}_2 &= \frac{\upsilon_2^2\left(2\mu'+\mu_3\right)-\upsilon_2^3-2\,\upsilon_2\mu'\mu_3+\left(\mu_3-\upsilon_2\right)\left(\mu'^2+\mu''^2\right)}{\upsilon_1\upsilon_3-\upsilon_2\upsilon_3+\upsilon_2^2-\upsilon_1\upsilon_2} \\ \tilde{b}_3 &= \frac{\upsilon_3^2\left(2\mu'+\mu_3\right)-\upsilon_3^3-2\,\upsilon_3\mu'\mu_3+\left(\mu_3-\upsilon_3\right)\left(\mu'^2+\mu''^2\right)}{\upsilon_1\upsilon_2-\upsilon_2\upsilon_3+\upsilon_3^2-\upsilon_1\upsilon_3} \end{aligned} \qquad (22)$$

To date, the typical chaotic attractors for this type of dynamical system have not been reported.

2.4. System F$\underline{\text{C}}$

This is the second case of hyphotetic dynamical system. The complete transformation matrix **T** is the same as for C$\underline{\text{C}}$ system, i.e. given by (16) together with auxiliary coefficients (17). Corresponding state equations of the new dynamical system can be obtained by symbolical examination of $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$, $\mathbf{T}^{-1}\mathbf{b}$ and $\mathbf{T}\mathbf{w}$ in the equation (7) with

$$\mathbf{A} = \begin{pmatrix} \upsilon_1+\upsilon_2 & -1 & \upsilon_1+\upsilon_2-2\mu' \\ \upsilon_1\upsilon_2 & 0 & \upsilon_1\upsilon_2-\mu'^2-\mu''^2 \\ 0 & 0 & \upsilon_3 \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} 2\mu'-\upsilon_1-\upsilon_2 \\ \mu'^2+\mu''^2-\upsilon_1\upsilon_2 \\ \mu_3-\upsilon_3 \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \tag{23}$$

The symbolical expressions of the mentioned matrix operations are too complicated and thus are not evaluated. The eigenvalues leading to some chaotic attractor are not given for the same reason as the previous case of dynamical system.

2.5. System $\underline{\text{C}}$F

The orthogonality of this system inside both outer segments suggests $\underline{\text{C}}$C system as the starting point. The partial transformation matrix **K** is the same as (8a) leading to the complete transformation given by (8b). The mathematical model is close to (9) with exception of elements of the vector $\mathbf{T}^{-1}\mathbf{b}$

$$\tilde{b}_1 = \mu_1+\mu_2+\mu_3-2\upsilon'-\upsilon_3-\frac{(\mu_3-\upsilon_3)(\upsilon_3-\mu_1)(\upsilon_3-\mu_2)}{(\upsilon_3-\upsilon')^2+\upsilon''^2}$$

$$\tilde{b}_3 = \frac{(\mu_3-\upsilon_3)(\upsilon_3-\mu_1)(\upsilon_3-\mu_2)}{(\upsilon_3-\upsilon')^2+\upsilon''^2} \tag{24}$$

$$\tilde{b}_2 = \frac{\mu_1(\mu_2+\mu_3)+\mu_2\mu_3-\delta-2\upsilon'\upsilon_3-(\upsilon'+\upsilon_3)b_1-2\upsilon'b_3}{\upsilon''}$$

And again Fig. 7 and Fig. 8 are given in order to numerically proove that DHA and DFA exist.

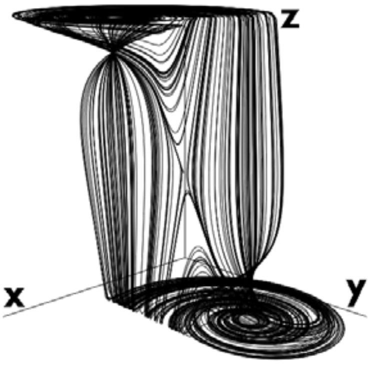

Fig. 7 DHA generated by $\underline{\text{C}}$F dynamical system.

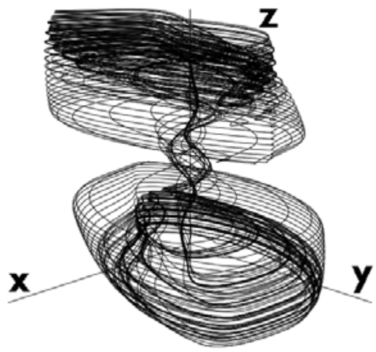

Fig. 8 DFA generated by $\underline{\text{C}}$F dynamical system.

2.6. System C$\underline{\text{F}}$

The approach to find this type of the dynamical system is the same as holds for the type C$\underline{\text{C}}$ system. To proove the universality of chosen process the complex decomposed and block diagonal model [4] has been accepted as the initial system. Such choice leads to the transformation matrix

$$\mathbf{T} = \begin{pmatrix} \dfrac{\mu_1-\mu_2}{\xi_2(\psi^{-1}\upsilon''-\upsilon'+\mu_1)} & 0 & 0 \\ 1/\xi_2 & \dfrac{(\mu_2-\mu_3)(\mu_1-\upsilon')}{\xi_3(\mu_3-\upsilon_3)\psi\upsilon''} & 0 \\ \dfrac{\xi_1+(\mu_1-\mu_2)(\mu_3-\upsilon_3)}{\xi_2} & 1/\xi_3 & 1 \end{pmatrix}, \tag{25}$$

where ψ is the optimization constant (provides minimization of eigenvalue sensitivity to the changes of the system parameters) and can be fixed on a single value ψ=1 and

$$\xi_1 = \frac{\upsilon''^2(\mu_3-\upsilon_3)}{\mu_1-\upsilon'}+(\mu_3-\upsilon_3)\psi\upsilon'',$$

$$\xi_2 = \sqrt{\left(\frac{\mu_1-\mu_2}{\psi^{-1}\upsilon''-\upsilon'+\mu_1}\right)^2+1+\left(\frac{\xi_1+(\mu_3-\upsilon_3)(\mu_1-\mu_2)}{(\psi^{-1}\upsilon''-\upsilon'+\mu_1)(\mu_1-\mu_3)}\right)^2},$$

$$\xi_3 = \sqrt{1+\left(\frac{(\mu_2-\mu_3)(\mu_1-\upsilon')}{(\mu_3-\upsilon_3)\psi\upsilon''}\right)^2}. \tag{26}$$

Having a nonsingular matrix **T** its inversion can be computed symbolically

$$\mathbf{T}^{-1} = \begin{pmatrix} \dfrac{\xi_2(\psi^{-1}\upsilon''-\upsilon'+\mu_1)}{\mu_1-\mu_2} & 0 & 0 \\ \dfrac{\xi_3(\psi^{-1}\upsilon''-\upsilon'+\mu_1)(\mu_3-\upsilon_3)\psi\upsilon''}{(\mu_1-\mu_2)(\mu_2-\mu_3)(\mu_1-\upsilon')} & \dfrac{\xi_3(\mu_3-\upsilon_3)\psi\upsilon''}{(\mu_2-\mu_3)(\mu_1-\upsilon')} & 0 \\ \dfrac{1}{\xi_2\xi_3\xi_4}\left(1-\dfrac{\xi_5}{(\mu_3-\upsilon_3)\psi\upsilon''}\right) & \dfrac{(\upsilon_3-\mu_3)\psi\upsilon''}{(\mu_2-\mu_3)(\mu_1-\upsilon')} & 1 \end{pmatrix}, \tag{27}$$

where additional constants are

$$\xi_4 = \frac{(\mu_1-\mu_2)(\mu_2-\mu_3)(\mu_1-\upsilon')}{\xi_2\xi_3(\psi^{-1}\upsilon''-\upsilon'+\mu_1)(\mu_3-\upsilon_3)\upsilon''\psi},$$

$$\xi_5 = (\mu_2-\mu_3)(\mu_1-\upsilon')[\xi_1+(\mu_1-\mu_2)(\mu_3-\upsilon_3)]. \tag{28}$$

The state equations of new dynamical system can be obtained by symbolical examination of $\mathbf{T}^{-1}\mathbf{AT}$, $\mathbf{T}^{-1}\mathbf{b}$ and $\mathbf{Tw}$ in the equation (6). Corresponding formulas are not given due to the above mentioned difficul-ties. Shape of the DHA generated by this system is also shown in Fig. 9. As required for this class of dynamical systems DFA can be also generated, see Fig. 10.

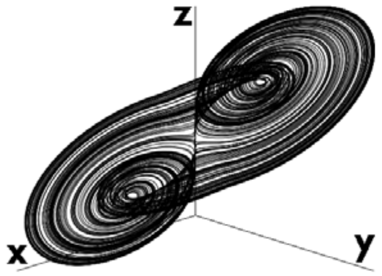

Fig. 9 DHA generated by C$\underline{\text{F}}$ dynamical system.

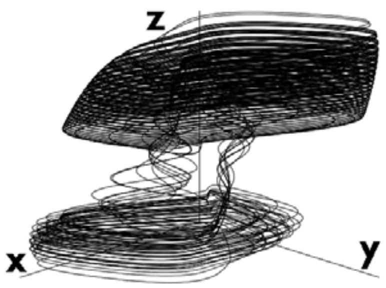

Fig. 10 DFA generated by C$\underline{\text{F}}$ dynamical system.

2.7. System $\underline{\text{F}}$F

Another hypothetical example of the examined class of dynamical systems have three real distinct eigenvalues in outer (orthogonal eigenvectors) and both inner segments of the vector field. Assume the vector **w** is exactly the same as for the system $\underline{\text{F}}$C. Thus the partial and complete transformation and its inverse are given by (20) and vector $\mathbf{T}^{-1}\mathbf{b}$ is

$$\tilde{b}_1 = \frac{\upsilon_1^2(\mu_1+\mu_2+\mu_3)-\upsilon_1(\upsilon_1^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3)+\mu_1\mu_2\mu_3}{\upsilon_1(\upsilon_1-\upsilon_2-\upsilon_3)+\upsilon_2\upsilon_3}$$

$$\tilde{b}_2 = \frac{\upsilon_2^2(\mu_1+\mu_2+\mu_3)-\upsilon_2(\upsilon_2^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3)+\mu_1\mu_2\mu_3}{\upsilon_2(\upsilon_2-\upsilon_1-\upsilon_3)+\upsilon_1\upsilon_3}$$

$$\tilde{b}_3 = \frac{\upsilon_3^2(\mu_1+\mu_2+\mu_3)-\upsilon_3(\upsilon_3^2+\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3)+\mu_1\mu_2\mu_3}{\upsilon_3(\upsilon_3-\upsilon_1-\upsilon_2)+\upsilon_1\upsilon_2}$$

The eigenvalues leading to some chaotic trajectory for this system are not known yet.

2.8. System F$\underline{\text{F}}$

Dynamical system with this geometry represents the last instance of hypothetic system in the sense that numerical values of the eigenvalues leading to chaos are not known so far. Assume the standard form of vector $\mathbf{w}=(1\ \ 1\ \ 1)^{\mathrm{T}}$ taken as a linear combi-nation of all state variables and vector **b** given by (27). Jordan form of the matrix $\mathbf{A}+\mathbf{bw}^{\mathrm{T}}$ implies that individual columns of the complete transformation **T** are normalized right eigenvectors associated with three distinct eigenvalues $\mu_{1,2,3}$

$$\mathbf{T} = \begin{pmatrix}\mathbf{T}_1 & \mathbf{T}_2 & \mathbf{T}_3\end{pmatrix},$$

$$\mathbf{T}_1 = \begin{pmatrix} 1 \\ \dfrac{(\upsilon_1-\upsilon_3)\left[\upsilon_2^2+\mu_1\mu_2-\upsilon_2(\mu_1+\mu_2)\right]}{(\upsilon_3-\upsilon_2)\left[\upsilon_1^2+\mu_1\mu_2-\upsilon_1(\mu_1+\mu_2)\right]} \\ \dfrac{(\upsilon_2-\upsilon_1)\left[\upsilon_3^2+\mu_1\mu_2-\upsilon_3(\mu_1+\mu_2)\right]}{(\upsilon_3-\upsilon_2)\left[\upsilon_1^2+\mu_1\mu_2-\upsilon_1(\mu_1+\mu_2)\right]} \end{pmatrix},$$

$$\mathbf{T}_2 = \begin{pmatrix} \dfrac{(\upsilon_3-\upsilon_2)\left[\upsilon_1^2+\mu_1\mu_3-\upsilon_1(\mu_1+\mu_3)\right]}{(\upsilon_1-\upsilon_3)\left[\upsilon_2^2+\mu_1\mu_3-\upsilon_2(\mu_1+\mu_3)\right]} \\ 1 \\ \dfrac{(\upsilon_2-\upsilon_1)\left[\upsilon_3^2+\mu_1\mu_3-\upsilon_3(\mu_1+\mu_3)\right]}{(\upsilon_1-\upsilon_3)\left[\upsilon_2^2+\mu_1\mu_3-\upsilon_2(\mu_1+\mu_3)\right]} \end{pmatrix}, \tag{30}$$

$$\mathbf{T}_3 = \begin{pmatrix} \dfrac{(\upsilon_2-\upsilon_3)\left[\upsilon_1^2+\mu_2\mu_3-\upsilon_1(\mu_2+\mu_3)\right]}{(\upsilon_1-\upsilon_2)\left[\upsilon_3^2+\mu_2\mu_3-\upsilon_3(\mu_2+\mu_3)\right]} \\ \dfrac{(\upsilon_3-\upsilon_1)\left[\upsilon_2^2+\mu_2\mu_3-\upsilon_2(\mu_2+\mu_3)\right]}{(\upsilon_1-\upsilon_2)\left[\upsilon_3^2+\mu_2\mu_3-\upsilon_3(\mu_2+\mu_3)\right]} \\ 1 \end{pmatrix}.$$

The corresponding columns of the inverse matrix can be computed by a substitution of the eigenvalues inside the following entries

$$\mathbf{T}^{-1} = \begin{pmatrix}\mathbf{T}_1^{-1} & \mathbf{T}_2^{-1} & \mathbf{T}_3^{-1}\end{pmatrix}$$

$$\mathbf{T}_1^{-1} = \begin{pmatrix} \dfrac{(\mu_3-\upsilon_2)(\upsilon_3-\mu_3)\left[\upsilon_1^2+\mu_1\mu_2-\upsilon_1(\mu_1+\mu_2)\right]}{(\mu_1-\mu_3)(\mu_2-\mu_3)(\upsilon_2-\upsilon_1)(\upsilon_1-\upsilon_3)} \\ \dfrac{(\mu_2-\upsilon_3)(\upsilon_2-\mu_2)\left[\upsilon_2^2+\mu_1\mu_3-\upsilon_2(\mu_1+\mu_3)\right]}{(\mu_1-\mu_2)(\mu_3-\mu_2)(\upsilon_1-\upsilon_2)(\upsilon_2-\upsilon_3)} \\ \dfrac{(\mu_1-\upsilon_3)(\mu_1-\upsilon_2)\left[\upsilon_3^2+\mu_2\mu_3-\upsilon_3(\mu_2+\mu_3)\right]}{(\mu_1-\mu_3)(\mu_2-\mu_1)(\upsilon_1-\upsilon_3)(\upsilon_3-\upsilon_2)} \end{pmatrix},$$

$$\mathbf{T}_2^{-1} = \begin{pmatrix} \dfrac{(\mu_3-\upsilon_3)(\upsilon_1-\mu_3)\left[\upsilon_1^2+\mu_1\mu_2-\upsilon_1(\mu_1+\mu_2)\right]}{(\mu_1-\mu_3)(\mu_2-\mu_3)(\upsilon_2-\upsilon_1)(\upsilon_1-\upsilon_3)} \\ \dfrac{(\mu_2-\upsilon_1)(\upsilon_3-\mu_2)\left[\upsilon_2^2+\mu_1\mu_3-\upsilon_2(\mu_1+\mu_3)\right]}{(\mu_1-\mu_2)(\mu_3-\mu_2)(\upsilon_1-\upsilon_2)(\upsilon_2-\upsilon_3)} \\ \dfrac{(\mu_1-\upsilon_1)(\mu_1-\upsilon_3)\left[\upsilon_3^2+\mu_2\mu_3-\upsilon_3(\mu_2+\mu_3)\right]}{(\mu_1-\mu_3)(\mu_2-\mu_1)(\upsilon_1-\upsilon_3)(\upsilon_3-\upsilon_2)} \end{pmatrix}, \tag{31}$$

$$\mathbf{T}_3^{-1} = \begin{pmatrix} \dfrac{(\mu_3-\upsilon_2)(\upsilon_1-\mu_3)\left[\upsilon_1^2+\mu_1\mu_2-\upsilon_1(\mu_1+\mu_2)\right]}{(\mu_1-\mu_3)(\mu_2-\mu_3)(\upsilon_2-\upsilon_1)(\upsilon_1-\upsilon_3)} \\ \dfrac{(\mu_2-\upsilon_1)(\upsilon_2-\mu_2)\left[\upsilon_2^2+\mu_1\mu_3-\upsilon_2(\mu_1+\mu_3)\right]}{(\mu_1-\mu_2)(\mu_3-\mu_2)(\upsilon_1-\upsilon_2)(\upsilon_2-\upsilon_3)} \\ \dfrac{(\mu_1-\upsilon_1)(\mu_1-\upsilon_2)\left[\upsilon_3^2+\mu_2\mu_3-\upsilon_3(\mu_2+\mu_3)\right]}{(\mu_1-\mu_3)(\mu_2-\mu_1)(\upsilon_1-\upsilon_3)(\upsilon_3-\upsilon_2)} \end{pmatrix}.$$

3. CONCLUSION

The mathematical models of dynamical systems derived in this paper are well suited for theoretical study and practical implementation. The eigenvalues belonging to class C can be considered as a quantifi-cation of oscillating subcircuit while class F suggests overdamped or undumped subcircuit. Moreover, the block diagonal nature of state matrices can simplify the synthesis of the final oscillator.

It should be noted that analogical classification of the dynamical systems can be done in the case of fourth-order dynamical systems. Such mathematical models can generate a very strange motion called hyperchaos. The motivation for making the classifi-cation given above is even much stronger then it is for the third-order dynamical systems. Searching for the novel forms of the state space representations is also up-to-date contribution to the synchronization and control topics as well.

The author believes that the uncovering a chaotic behavior associated to the hypothetic cases of the dynamical systems presented in this paper is only a matter of time. The key role in this may play the concept of the stochastic optimizations recently published in [5]. The most important thing is that these systems exists and can be numerically solved. The mathematical proove of chaos in the sense of Shilnikov´s theorems is the topic for future study.

If the reader is interested please do not hesitate to write to the author for the Mathcad script uncovering the geometry of all linear transformation of the state space coordinates mentioned above. All sets of the parameters leading to the evolution of the strange attractor can be also provided.

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BIOGRAPHY

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