THE COMPUTING PROCEDURE FOR THE MULTIPARAMETRIC EIGENPROBLEM IN MAX ALGEBRA

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ABSTRACT

Denote $a \oplus b = \max(a,b)$, and $a \otimes b = a+b$, for $a,b \in \mathbb{R}$ and extend this pair of operations to matrices and vectors in the same way as in conventional linear algebra, that is if $A=(a_{ij}), B=(b_{ij}), C=(c_{ij})$ are real matrices or vectors of compatible sizes then $C = A \otimes B$ if $c_{ij} = \bigoplus_k a_{ik} \otimes b_{kj}$ for all i,j. For any $n \times n$ matrix $A=(a_{ij})$ and for arbitrary sequence of real parameters $\alpha = (\alpha_1, \ldots, \alpha_p)$, $p \leq n$, the problem of finding all $x(\alpha)$ and $\lambda(A(\alpha))$ satisfying $A(\alpha) \otimes x(\alpha) = \lambda(A(\alpha)) \otimes x(\alpha)$ is studied. The problem is called Multiparametric Eigenproblem *(in short: MPE). We introduce some properties of the general MPE and we suggest a pseudopolynomial $O(pn^5)$ algorithm for computing all eigenvectors of MPE.*

Keywords: *extremal eigenvalue, eigenvector*

1. INTRODUCTION

Large number of systems, in which the individual components are moving from event to event rather that varying continuously through time, can be represented by discrete-event systems. A characteristic of many such discrete-event systems is that any given component must wait before proceeding to its next event, until certain other components will have completed their current event. Significant effort has been made to build up a theory similar to that of linear algebra, which would properly describe the behavior of discrete-event systems. In particular, the systems of linear equations and the eigenproblem were intensively investigated. Cuninghame-Green [6] discussed the following eigenproblem in connection with a hypothetical industrial discrete-event system - a worker on an assembly line cannot begin a new assembly operation until, say, two interlocking sub-assemblies have arrived from different sources with independent production rates. A natural way of describing such system is to label the machines, e.g. by numbers $1,2,\ldots,n$, and to describe the interferences by recurrence relations

$$x_i(r+1) = \max\big(x_1(r)+a_{1i}, \ldots, x_n(r)+a_{ni}\big)$$

for $i \in \{1,2,\ldots,n\}$. The symbol $x_i(r)$ in the above formula denotes the starting time of the rth cycle of machine i, and a_{ij} stands for the duration of the corresponding activity.

Let us denote $a \oplus b = \max(a,b)$, and $a \otimes b = a+b$ for $a,b \in \mathbb{R}$ and extend this pair of operations to matrices and vectors in the same way as in conventional linear algebra, that is if $A=(a_{ij}), B=(b_{ij}), C=(c_{ij})$ are real matrices or vectors of compatible sizes then $C = A \otimes B$ if $c_{ij} = \bigoplus\limits_k a_{ik} \otimes b_{kj}$ for all i,j.

By generalization of the discrete-event system and the eigenproblem mentioned above, we obtain a description of the stable system, in which the interval between the beginnings of consecutive cycles on every machine is some constant λ

$$x(r+1) = A(r) \otimes x(r) \text{ and } x(r+1) = \lambda \otimes x(r)$$

in matrix notation

$$A \otimes x(r) = \lambda \otimes x(r)$$

The aim of this paper is to study this model in situation when changes of activity durations a_{jk} occur. We analyze the case when the entries of a given matrix $A=(a_{ik})$ are given waggly, i.e. the values a_{ik} can be changed by the parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ to $a_{ik} + \alpha_k$ for all i and for $k = 1,2,\ldots,p$.

2. DEFINITIONS AND PRELIMINARY RESULTS

Let $(G, \otimes, \leq)$ be a linearly ordered, commutative group with the neutral element $e=0$. We suppose that G is radicable, i.e. for every integer $t \geq 1$ and for every $a \in G$, there exists a (unique) element $b \in G$ such that $b^t = a$, the element will be denoted as $b = a^{1/t}$.

Throughout the paper, $n \geq 1, m \geq 1$ are given integers. The set of $n \times m$ matrices over G is denoted by $G(n,m)$. We introduce a further binary operation $\oplus$ on G by the formula

$$a \oplus b = \max(a,b) \quad \text{for all } a,b \in G$$

The triple $(G, \oplus, \otimes)$ is called *max-algebra*. If $G = \mathbb{R}$ and the group $(G, \otimes, \leq)$ is the additive group of real numbers, then $(G, \oplus, \otimes)$ is called *max-plus algebra* (often used in applications). The operations $\oplus, \otimes$ are extended to the matrix algebra over G by the direct analogy to the conventional linear algebra. Further, we extend G by a new element $-\infty$, denoting $G \cup \{-\infty\}$ by $\bar{G}$ and extending $\otimes$ and $\leq$ to $\bar{G}$: $a \otimes -\infty = -\infty \otimes a = -\infty$ and $-\infty < a$ for all $a \in G$. The symbol $\mathrm{diag}(d_1, d_2, \ldots, d_n)$ denotes the matrix D with diagonal elements equal to $d_1, d_2, \ldots, d_n \in \bar{G}$ and off-diagonal elements equal to $-\infty$. Matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ will be called *diagonal* if $d_1, d_2, \ldots, d_n \in G$, i.e. all diagonal elements are finite (none of them is equal to $-\infty$).

Aim of this paper is to give a description of the eigenvalues and of the eigenspace of a matrix in max-algebra $(G, \oplus, \otimes)$, with respect to a given sequence of parameters $\alpha = (\alpha_1, \ldots, \alpha_p)$ in G. Below we introduce the necessary notation and recall some preliminary results.

Let $N = \{1,2,\ldots,n\}$ and let C_n be the set of all cyclic permutations defined on nonempty subsets of N. For a cyclic permutation $\sigma = (i_1, i_2, \ldots, i_l) \in C_n$ and for $A \in G(n,n)$ we denote the length of σ by $l = l(\sigma)$ and define

$$w_A(\sigma) = a_{i_1 i_2} \otimes a_{i_2 i_3} \otimes \cdots \otimes a_{i_l i_1}, \quad \mu_A(\sigma) = w_A(\sigma)^{1/l(\sigma)}$$

$$\lambda(A) = \bigoplus_{\sigma \in C_n} \mu_A(\sigma)$$

where $\bigoplus$ denotes the iterated use of the operation $\oplus$.

The *eigenproblem* in max-algebra is formulated as follows. Given $A \in G(n,n)$, find $x \in G(n,1)$ and $\lambda(A) \in G$ satisfying

$$A \otimes x = \lambda(A) \otimes x$$

This problem was treated by several authors during the sixties, c.g. [5, 8], survey of the results concerning this and similar eigenproblems can be found in [19, 20].

The *ℓ-parametric eigenproblem* in max-algebra, which was studied in [18], is defined similarly as the eigenproblem formulated above, but the entries in the first ℓ columns of A are modified by adding a common parameter α.

The *Multiparametric eigenproblem* (in short: MPE) in max-algebra is defined as follows: for a given square matrix $A = (a_{ij}) \in G(n,n)$ and for arbitrary sequence of parameters $\alpha = (\alpha_1, \ldots, \alpha_p)$ in G of length $p \leq n$, define the matrix $A(\alpha) = \big(a_{ij}(\alpha)\big) \in G(n,n)$ by

$$a_{ij}(\alpha) = \begin{cases} a_{ij} + \alpha_j & \text{if } j \leq p \\ a_{ij} & \text{otherwise} \end{cases}$$

Then find $x(\alpha) \in G(n,1)$ and $\lambda(A(\alpha)) \in G$ satisfying

$$A(\alpha) \otimes x(\alpha) = \lambda(A(\alpha)) \otimes x(\alpha)$$

The value $\lambda(\alpha)$ is called *multiparametric eigenvalue*, and $x(\alpha)$ is called *multiparametric eigenvector* of A. The MPE case for $p = 2$ (so called *biparametric eigenproblem*) was solved in [11].

The symbol $D_A = (V,E)$ stands for a complete, arc-weighted digraph associated with A. The node set of D_A is N, and the weight of any arc (i,j) is a_{ij}. Throughout the paper, by a cycle in the digraph we mean an elementary cycle or a loop, and by path we mean a nontrivial elementary path, i.e. an elementary path containing at least one arc. We will use the same notation, as well as the concept of weight, both for cycles and cyclic permutations on subsets of N. A cycle $\sigma \in C_n$ is *optimal*, if $\mu_A(\sigma) = \lambda(A)$, a node in D_A is called an *eigennode* if it is contained in at least one optimal cycle. Notation E_A is used for the set of all eigennodes in D_A.

Theorem 2.1 [6] *Each square matrix has at most one eigenvalue. If G is radicable, then every square matrix A has exactly one eigenvalue. This unique eigenvalue is equal to the maximal average weight of cycles in D_A.*

According to the notation introduced above, the unique eigenvalue of A is denoted as $\lambda(A)$ in what follows.

Theorem 2.2 [6] *Let G be radicable, $A \in G(n,n)$ and $c \in G$. Then*

$$\lambda(c \otimes A) = c \otimes \lambda(A)$$

The problem of finding the eigenvalue $\lambda(A)$ is also called the *maximum cycle mean problem* and it has been studied by several authors [1–8, 10, 13, 15–18]. Various algorithms for solving this problem are known, that of Karp [13] having the best worst-case performance $O(n^3)$ and Howard's algorithm [12] of unproved computational complexity showing excellent algorithmic performance. For $B \in G(n,n)$ we denote by $\Delta(B)$ the matrix $B \oplus B^2 \oplus \ldots \oplus B^n$ where B^s stands for the s-fold iterated product $B \otimes B \otimes \ldots \otimes B$.

Let $A_\lambda = \lambda(A)^{-1} \otimes A$. (The upper index -1 denotes the inverse element of $\lambda(A)$ in the sense of the group operation $\otimes$). It is shown in [6] that the matrix $\Delta(A_\lambda)$ contains at least one column, the diagonal element of which is e. Every such column is an eigenvector of the matrix A, and it is called a *fundamental eigenvector* of the matrix A. The set of all fundamental eigenvectors will be denoted by F_A and its cardinality is denoted by $q = |F_A|$. We say that $x, y \in F_A$ are equivalent if $x = c \otimes y$ for some $c \in G$. In what follows $s(A)$ denotes the set of all eigenvectors of A, so called *eigenspace* of A.

Theorem 2.3 [6] *Let $A \in G(n,n)$. Then*

$$s(A) = \left\{ \bigoplus_{i=1}^{q} c_i \otimes g_i;\ c_i \in G,\ g_i \in F_A,\ i = 1,2,\ldots,q \right\}$$

It follows from the definition of equivalent fundamental eigenvectors that the set F_A in Theorem 2.3 can be replaced by any maximal set F'_A of fundamental eigenvectors such that no two of them are equivalent. Every such set F'_A will be called a complete set of generators (of the eigenspace $s(A)$).

Theorem 2.4 [6] *Let $g_1, g_2, \ldots, g_n$ denote the columns of the matrix $\Delta(A_\lambda)$. Then*

1. *$j \in E_A$ if and only if $g_j \in F_A$*
2. *g_i, g_j are equivalent members of F_A if and only if the eigennodes i, j are contained in a common optimal cycle*

Let be $\Delta(A_\lambda) = (\xi_{ij})$. It follows from the definition of $\Delta(A_\lambda)$ that ξ_{ij} is the weight of the heaviest path from i to j in D_A. Hence, $\Delta(A_\lambda)$ can be computed in $O(n^3)$ operations using the Floyd-Warshall algorithm [14]. By trivial search and comparisons one can then find a complete set of fundamental eigenvectors among the columns of $\Delta(A_\lambda)$, using at most $O(n^3)$ operations. The next assertion follows straightforwardly from the definition of $\Delta(A_\lambda)$.

Theorem 2.5 *Let $d \in G$, $A \in G(n,n)$ and $D = \mathrm{diag}(d, \ldots, d)$. Then $\Delta(A_\lambda) = \Delta((A \otimes D)_\lambda)$.*

3. MULTIPARAMETRIC EIGENVALUE

In this section we investigate the *multiparametric maximum cycle mean* for $A(\alpha)$, where $A=(a_{ij})\in G(n,n)$ is a given matrix and $\alpha=(\alpha_1,\ldots,\alpha_p)$ is an arbitrary sequence in G, of length $|\alpha|=p\le n$. In the rest of the paper, the numbers n,p and the matrix A are arbitrary, but fixed. We shall use the notation $N=\{0,1,\ldots,n\}$, $P=\{0,1,\ldots,p\}$. According to Theorem 2.2, we may assume without any loss of generality that the given matrix A has the property $\lambda(A)=0$. We shall only consider the case $G=\mathbb{R}$, which allows interesting geometrical interpretation, however, the reasoning in the general case is analogical, and the described algorithms work in any max-algebra G.

Matrix A can be written in the following block diagonal form

$$A=\begin{pmatrix} B & . \\ . & C \end{pmatrix}$$

where B and C are $p\times p$ and $(n-p)\times(n-p)$ square submatrices of A, respectively, and the dots denote submatrices of corresponding dimensions. The next theorem describes an easily provable lower bound for $\lambda(A(\alpha))$.

Theorem 3.1 *If $\alpha_1,\ldots,\alpha_p\ge 0$ then*

$$\lambda(A(\alpha))\ge\max\big(\lambda(B(\alpha)),\lambda(C)\big)$$

Proof. By Theorem 2.1, the value $\lambda(A(\alpha))$ is the maximum cycle mean in $D_{A(\alpha)}$, while $\lambda(B(\alpha))$ and $\lambda(C)$ are the maximum cycle means in $D_{B(\alpha)}$ and D_C, respectively. The inequality follows directly from the fact that $D_{B(\alpha)}$ and D_C are subgraphs of $D_{A(\alpha)}$.
□

Our aim is to compute the exact value of $\lambda(A(\alpha))$ for arbitrary values in the parameter vector $\alpha=(\alpha_1,\alpha_2,\ldots,\alpha_p)$. Therefore, we have to consider not only the cycles in $D_{B(\alpha)}$, (with all vertices in P), or the cycles in D_C (with vertices in $N-P$), but also the cycles visiting both sets P and $N-P$. Such 'mixed' cycles will be classified according to their intersection with the set of parameterized columns.

We introduce the following notation: for any cycle (permutation) $\sigma=(i_1,i_2,\ldots,i_s)\in C_n$ we denote $P(\sigma)=P\cap\{i_1,i_2,\ldots,i_s\}$, $N(\sigma)=(N-P)\cap\{i_1,i_2,\ldots,i_s\}$. Clearly, $P(\sigma)$ and $N(\sigma)$ are disjoint, and $\{i_1,i_2,\ldots,i_s\}=P(\sigma)\cup N(\sigma)\subseteq P(\sigma)\cup(N-P)$, which implies $|P(\sigma)|\le s\le|P(\sigma)|+n-p$.

For any set $J=\{j_1,j_2,\ldots,j_k\}\subseteq P$ and for any $s\in N$, $s\ge|J|$, we denote by $C^J_{n,s}$ the set of all cyclic permutations $\sigma\in C_n$, of length s, fulfilling the equality $P(\sigma)=J$. We define

$$m^J_s=\max\left\{\mu_A(\sigma);\sigma\in C^J_{n,s}\right\},$$

$$M^J_s(\alpha)=m^J_s+\frac{\alpha_{j_1}+\cdots+\alpha_{j_k}}{s}$$

It is easy to see that $M^J_s(\alpha)=\max\left\{\mu_{A(\alpha)}(\sigma);\sigma\in C^J_{n,s}\right\}$ for any $J\subseteq P$. Clearly, for $J=\emptyset$ we have $m^\emptyset_s=M^\emptyset_s(\alpha)$. For every $J\subseteq P$ and for every $v\in N$, $|J|\le v\le|J|+n-p$ we define the sets

$$\mathscr{P}^J_{\ge}(v)=\{\alpha\in\mathbb{R}^p;\ M^J_v(\alpha)\ge$$

$$\max\big\{M^E_s(\alpha);E\subseteq P,|E|\le s\le|E|+n-p\big\}\}$$

$$\mathscr{P}^J_{>}(v)=\{\alpha\in\mathbb{R}^p;\ M^J_v(\alpha)>\max\big\{M^E_s(\alpha);$$

$$E\subseteq P,|E|\le s\le|E|+n-p,(s,E)\ne(v,J)\big\}\}$$

The well-known notion of a convex polyhedron in a p-dimensional linear space $\mathbb{R}^p$ is defined as the set of all solutions to a finite system of linear inequalities with p real variables. The coordinates of all vertices of a given convex polyhedron may be found by a process called 'vertex enumeration'. Next two assertions follow from the above definitions.

Theorem 3.2 *The set $\mathscr{P}^J_{\ge}(v)$ is a convex polyhedron in $\mathbb{R}^p$, for any $J\subseteq P$, $v\in N$, $|J|\le v\le|J|+n-p$.*

Theorem 3.3 *Let $\alpha\in\mathscr{P}^J_{\ge}(v)$. Then $\lambda(A(\alpha))=M^J_v(\alpha)$.*

3.1. Computing procedure

To solve the multiparametric maximum cycle mean problem completely, we need an efficient way for computing all values m^J_s. The computation can be done by the recursive process described below.

We shall use the following notation: A_P will denote the $n\times n$ matrix which arises from the matrix A by replacing all entries of the first p rows by $-\infty$. Further, for any $j\in P$ we denote by $A^{\{j\}}$ the matrix created from A by replacing all entries, except those of the jth row, by $-\infty$.

We compute matrices B^J_s for all $J\subseteq P$, $s\in N$, $|J|\le s\le|J|+n-p$. The computation proceeds by recursion on $|J|$. For $|J|=0$ we have $J=\emptyset$ and we put

$$B^\emptyset_0=\mathrm{diag}(e,e,\ldots,e)\quad\text{for } s=0$$

$$B^\emptyset_s=A_PB^\emptyset_{s-1}\quad\text{for } 1\le s\le n-p$$

For $|J|=1$ we have one-element subsets $J=\{j\}$, $j=1,2,\ldots,p$. For all j and for $1\le s\le 1+n-p$ we put

$$B^{\{j\}}_s=\max\left\{B^\emptyset_rA^{\{j\}}B^\emptyset_{s-1-r};r=0,1,\ldots,s-1\right\}$$

If $1<k\le p$ and the matrices B^J_s with $J\subseteq P$, $s\in N$, $|J|\le s\le|J|+n-p$ are defined for all subsets of P of the cardinality $<k$, then for all $J\subseteq P$ with $|J|=k$ and for all $s\in N$, $k\le s\le k+n-p$ we put

$$B^J_s=\max\left\{B^\emptyset_rA^{\{j\}}B^{J-\{j\}}_{s-1-r};j\in J,r=0,1,\ldots,s-k\right\}.$$

Theorem 3.4 *Let $J\subseteq P$, $|J|\le s\le|J|+n-p$. Then, for any $j\in J$,*

$$m^J_s=\frac{(b^J_s)_{jj}}{s},\quad M^J_s=\frac{(b^J_s)_{jj}+\sum\{\alpha_h;h\in J\}}{s}$$

Proof. It is easy to see that the elements $(b^J_s)_{ik}$ of the matrix B^J_s computed in the above recursive process are equal to the maximal weight of a path from i to k, of length s, containing exactly one edge starting in each vertex $j\in J$ and not containing edges starting in vertices belonging to $P-J$. Hence the diagonal inputs in the columns of B^J_s corresponding to vertices in J contain the maximal weight of

a cycle, of length s, visiting J in each of its vertices exactly once, and non-visiting vertices in $P-J$. The assertion of the theorem is a direct consequence of this observation.
□

When p is fixed, then the described recursion is polynomial in n. With increasing p, the computational complexity of the multiparametric eigenproblem grows exponentially. Hence, the problem is only efficiently solvable, when the number of parameters is not very large.

Theorem 3.5 *The values m_s^J for all $J \subseteq P$ and for all $s \in N$ with $|J| \leq s \leq |J|+n-p$, can be computed $O\left(n^3(n-p+1)^2 p^2 2^p\right)$ time, i.e. in $O(n^5)$ time when the number of parameters p is fixed.*

Proof. For any given $J \subseteq P$ and $s \in N$, the computation of the matrix B_s^J requires at most $(s-k+1)k$ maximum operations (in notation $k = |J|$), and twice as many matrix multiplications. By inequalities $s-k+1 \leq n-p+1$, $k \leq p$ we get the complexity $O\left(n^3(n-p+1)p2^p\right)$ for every pair (J,s). As the computation is performed in $p+1$ recursive steps for $k = 0,1,2,\ldots,p$, $J \subseteq P$ and $s \in N$ with $k \leq s \leq k+n-p$, the total computational complexity of the recursion is $O\left(n^3(n-p+1)^2 p^2 2^p\right)$.
□

4. MULTIPARAMETRIC EIGENVECTOR

In this section we investigate how the eigenvectors in the multiparametric problem depend on parameters $\alpha_1,\ldots,\alpha_p$. The following theorem describes the simplest situation, when the dimension of the eigenspace of $A(\alpha)$ is 1.

Theorem 4.1 *Let $\alpha \in \mathscr{P}_{>}^J(v)$. Then $|F'_{A(\alpha)}| = 1$.*

The coordinates all eigenvectors of matrix $A(\alpha)$ are given by columns of the metric matrix $\Delta(A_\lambda(\alpha))$.

Theorem 4.2 *Let $\alpha \in \mathscr{P}_{>}^J(v)$. Then for every $j \in J$, the jth column $\xi_j(\alpha)$ of $\Delta(A_\lambda(\alpha))$ is an eigenvector of $A(\alpha)$ and its coordinates are given by the formula*

$$\xi_{ij}(\alpha) = \max\left\{\left(b_s^E\right)_{ij} + \sum\left\{\alpha_h; h \in E\right\} - sM_v^J;\right.$$

$$\left. E \subseteq P, |E| \leq s \leq |E|+n-p\right\}$$

for $i = 1,2,\ldots,n$.

Proof. According to Theorem 3.3, all inputs in $A_\lambda(\alpha)$ are equal to the inputs in $A(\alpha)$, diminished by $\lambda(A(\alpha)) = M_v^J(\alpha)$. By the definition of the metric matrix, each coordinate $\xi_{ij}(\alpha)$ is equal to the maximal weight of an elementary path p from i to j in the digraph D associated with $A_\lambda(\alpha)$. As p is elementary, there is a subset $E \subseteq P$ such that the path p contains exactly one edge starting in every vertex $h \in E$, and contains no edges starting in vertices in $P-E$. Hence the weight $w_\alpha(p)$ computed in $A_\lambda(\alpha)$ is equal to the weight $w(p)$ computed in A, plus the sum of all α_h with $h \in E$, minus $l(p)$ times M_v^J. As the maximum of weights $w(p)$, for fixed E and fixed length $l(p) = s$, is equal to $\left(b_s^E\right)_{ij}$, the above formula gives the desired result.
□

Remark 1 The formula in Theorem 4.2 gives the values $\xi_{ij}(\alpha)$ for fixed $j \in J$ in time $O(n(n-p+1)2^p)$.

4.1. Procedure Multiparameter

In view of Theorems 4.1 and 4.2, the eigenvectors of a given matrix with parameters in the first p columns can be computed by the following procedure.

Procedure **Multiparameter**
Input: Given matrix $A \in \mathbb{R}(n,n)$, $p \leq n$
Output: $\lambda(A(\alpha))$, $\xi_{ij}(\alpha)$, for every $\alpha \in \mathbb{R}^p$, $i \in N$, $j \in J \subseteq P$
1. *Compute* m_v^J for all $J \subseteq P$, $v \in N$, $|J| \leq v \leq n$
2. *Compute* $\xi_{ij}(\alpha)$, $i \in N$, $j \in J$ for every $\alpha \in \mathscr{P}_{>}^J(v)$

Theorem 4.3 *Procedure* **Multiparameter** *works correctly and terminates in $O\left(n^3(n-p+1)^2 2^{2p}\right)$ time, i.e. in $O(n^5)$ time when the number of parameters p is fixed.*

Proof. Part 1. It has been shown in the proof of Theorem 3.5 that the matrices B_v^J, and the values m_v^J for all $J \subseteq P$ and for all $v \in N$ with $|J| \leq v \leq |J|+n-p$, can be computed in $O\left(n^3(n-p+1)^2 p^2 2^p\right)$ time.

Part 2. It follows from Remark 1 that a procedure which computes $\xi_{ij}(\alpha)$ for fixed $J \subseteq P$ and fixed $j \in J$ has the complexity $O(n(n-p+1)2^p)$. Since we aim to compute one eigenvector $\xi_{ij}(\alpha)$ for every $J \subseteq P$ and $v \in N$, $|J| \leq v \leq |J|+n-p$, we have the computational complexity in Part 3 equal to $O(n(n-p+1)2^p) \cdot O((n-p+1)2^p) = O\left(n(n-p+1)^2 2^{2p}\right)$.

Summarizing the computational complexity of Parts 1 and 2, we get the total complexity of **Multiparameter** procedure equal to $O\left(n^3(n-p+1)^2 2^{2p}\right)$.
□

5. SPECIAL CASE - THREEPARAMETRIC EIGENPROBLEM

In this section we discuss the special case, when only parameters in the first three columns are considered. For reader's convenience we avoid use of superfluous indices denoting the parameters by α,β,γ, instead of $\alpha_1,\alpha_2,\alpha_3$.

Procedure **Threeparameter**
Input: A given matrix $A \in \mathbb{R}(n,n)$
Output: $\lambda(A(\alpha,\beta,\gamma))$, $\xi_{ij}(\alpha,\beta,\gamma)$ for every $(\alpha,\beta,\gamma) \in \mathbb{R}^3$, $i \in N$, $j \in J \subseteq \{1,2,3\}$
1. *Compute* m_v^J for $J \in \{\emptyset,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$, $v \in N$
2. *Compute* $\xi_{ij}(\alpha,\beta,\gamma)$, $i \in N$, $j \in J$ for every $(\alpha,\beta,\gamma) \in \mathscr{P}_{>}^J(v)$.

Theorem 5.1 *Procedure* **Threeparameter** *works correctly and terminates in $O(n^5)$ time.*

Next example shortly demonstrates the work of the **Threeparameter** procedure on a parametric matrix with values $n = 3$, $p = 3$.

Example 1 Let matrices A and $A(\alpha,\beta,\gamma)$ have the following form

$$A = \begin{pmatrix} -6 & 0 & -4 \\ -2 & -3 & -2 \\ 2 & 0 & -1 \end{pmatrix}$$

$$A(\alpha,\beta,\gamma) = \begin{pmatrix} -6+\alpha & 0+\beta & -4+\gamma \\ -2+\alpha & -3+\beta & -2+\gamma \\ 2+\alpha & 0+\beta & -1+\gamma \end{pmatrix}$$

By recursion we get

$$M_1^{\{1\}} = -6+\alpha, \quad M_1^{\{2\}} = -3+\beta, \quad M_1^{\{3\}} = -1+\gamma$$

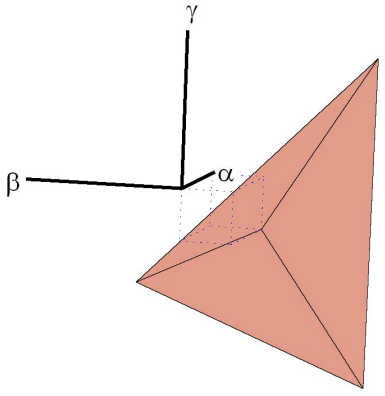
$$M_2^{\{1,2\}} = \frac{-2+\alpha+\beta}{2}, \quad M_2^{\{1,3\}} = \frac{-2+\alpha+\gamma}{2},$$

$$M_2^{\{2,3\}} = \frac{-2+\beta+\gamma}{2}, \; M_3^{\{1,2,3\}} = \frac{\alpha+\beta+\gamma}{3}.$$

The case $\lambda(A(\alpha,\beta,\gamma)) = M_1^{\{1\}} = -6+\alpha$ will be considered in detail. For the convex polyhedron $\mathscr{P}_{\geq}^{\{1\}}(1)$ we get a system of inequalities

$$-6+\alpha \geq -3+\beta \quad \Leftrightarrow \quad \alpha-\beta-3 \geq 0 \qquad (1)$$
$$-6+\alpha \geq -1+\gamma \quad \Leftrightarrow \quad \alpha-\gamma-5 \geq 0 \qquad (2)$$
$$-6+\alpha \geq \frac{-2+\alpha+\beta}{2} \quad \Leftrightarrow \quad \alpha-\beta-10 \geq 0 \qquad (3)$$
$$-6+\alpha \geq \frac{-2+\alpha+\gamma}{2} \quad \Leftrightarrow \quad \alpha-\gamma-10 \geq 0 \qquad (4)$$
$$-6+\alpha \geq \frac{-2+\beta+\gamma}{2} \quad \Leftrightarrow \quad 2\alpha-\beta-\gamma-10 \geq 0 \qquad (5)$$
$$-6+\alpha \geq \frac{\alpha+\beta+\gamma}{3} \quad \Leftrightarrow \quad 2\alpha-3\beta-3\gamma-18 \geq 0 \qquad (6)$$

The inequalities (1), (2) and (5) may be neglected, because they follow directly from (3) and (4). Hence, the polyhedron $\mathscr{P}_{\geq}^{\{1\}}(1)$ is the intersection of three half-spaces determined by inequalities $\alpha-\beta \geq 10$, $\alpha-\gamma \geq 10$ and $2\alpha-3\beta-3\gamma \geq 18$. The shape of the polyhedron is partly shown in the following picture (the polyhedron is unbounded in the direction to negative values of α and β).



For computing the coordinates of an eigenvector $x(\alpha,\beta,\gamma) = (\xi_{11},\xi_{21},\xi_{31})$ parametrized by a triple $(\alpha,\beta,\gamma) \in \mathscr{P}_{\geq}^{\{1\}}(1)$ we shall use Theorem 4.2. We get general formulas for eigenvector coordinates

$$\xi_{11} = 0,$$
$$\xi_{21} = \max\{a_{21}+\alpha-(-6+\alpha), a_{22}+a_{21}+\alpha+\beta-$$
$$2(-6+\alpha), a_{23}+a_{32}+a_{21}+\alpha+\beta+\gamma-3(-6+\alpha)\}$$
$$\xi_{31} = \max\{a_{31}+\alpha-(-6+\alpha), a_{32}+a_{21}+\alpha+\gamma-$$
$$2(-6+\alpha), a_{33}+a_{32}+a_{21}+\alpha+\beta+\gamma-3(-6+\alpha)\}.$$

Inserting the numerical values from the matrix A we get the eigenvector coordinates in the form

$$\xi_{11} = 0$$
$$\xi_{21} = \max\{4, 7-\alpha+\beta, 14-2\alpha+\beta+\gamma\}$$
$$\xi_{31} = \max\{8, 10-\alpha+\gamma, 15-2\alpha+\beta+\gamma\}.$$

For any triple of parameters $(\alpha,\beta,\gamma) \in \mathscr{P}_{>}^{\{1\}}(1)$, e.g. for $(6,-6,-6)$, the eigenspace consists of a single fundamental vector with coordinates equal to $(0,4,8)^T$.

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