AN APPLICATION OF STOCHASTIC INTEGRAL EQUATIONS TO ELECTRICAL NETWORKS

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ABSTRACT

In this paper we present an application of the Itô stochastic calculus to the problem of modelling inductor-resistor electrical circuits. The deterministic model of the circuit is replaced by a stochastic model by adding a noise term in the source. We also consider the case when both the source and the resistance are random. The analytic solutions of the resulting stochastic integral equations are found using the multidimensional Itô formula. We also examined statistical estimates of the stochastic solutions. The programming language C#, a part of the new MS .NET platform, is used for numerical simulations.

Keywords: *stochastic differential equations, Itô formula, RL electrical circuit, C#*

1. INTRODUCTION

Modelling of physical systems by ordinary differential equations ignores stochastic effects. By incorporating random elements into the differential equations, a system of stochastic differential equations (SDEs) arises.

A general N-dimensional SDE can be written in vector form as

$$\mathrm{d}\mathbf{X}(t) = \mathbf{A}(t,\mathbf{X}(t))\,\mathrm{d}t + \sum_{j=1}^{M} \mathbf{B}^j(t,\mathbf{X}(t))\,\mathrm{d}W^j(t), \tag{1}$$

where $\mathbf{A} : \langle 0,T\rangle \times \mathbb{R}^N \to \mathbb{R}^N$ is a vector function, $\mathbf{B}^j$ represents the $j-$th column of the matrix function $\mathbf{B} : \langle 0,T\rangle \times \times \mathbb{R}^N \to \mathbb{R}^{N\times M}$ and $\mathrm{d}\mathbf{W}(t) = (\mathrm{d}W^1(t),\dots, \mathrm{d}W^M(t))$ is a column vector, where $W^1(t),\dots,W^M(t)$ are independent Wiener processes representing the noise. (A stochastic process $W(t)$ is called the Wiener process if it has independent increments, $W(0) = 0$ and $W(t)-W(s)$ distributed $N(0,t-s)$, $0 \leqq s < t$). The solution is a stochastic vector process $\mathbf{X}(t) = (X^1(t),\dots,X^N(t))$. By an SDE we understand in fact an integral equation

$$\mathbf{X}(t) = \mathbf{X}_0 + \int_{t_0}^{t} \mathbf{A}(s,\mathbf{X}(s))\,\mathrm{d}s + \sum_{j=1}^{M}\int_{t_0}^{t} \mathbf{B}^j(s,\mathbf{X}(s))\,\mathrm{d}W^j(s), \tag{2}$$

where the integral with respect to $\mathrm{d}s$ is the Lebesgue integral and the integrals with respect to $\mathrm{d}W^j(s)$ are stochastic integrals, called the Itô integrals (see [5]).

Although the Itô integral has some very convenient properties, the usual chain rule of classical calculus doesn't hold. Instead, the appropriate stochastic chain rule, known as Itô formula, contains an additional term, which, roughly speaking, is due to the fact that the stochastic differential $(\mathrm{d}W(t))^2$ is equal to $\mathrm{d}t$ in the mean square sense, i.e. $E[(\mathrm{d}W(t))^2] = \mathrm{d}t$, so the second order term in $\mathrm{d}W(t)$ should really appear as a first order term in $\mathrm{d}t$.

The multidimensional Itô formula. Let the stochastic process $\mathbf{X}(t)$ be a solution of the stochastic differential equation (1) for some suitable matrix functions $\mathbf{A},\mathbf{B}$ (see [5], p. 48).

Let $\mathbf{g}(t,\mathbf{x}) : (0,\infty)\times \mathbf{R}^N \to \mathbf{R}^P$ is a twice continuously differentiable function. Then

$$\mathbf{Y}(t) = \mathbf{g}(t,\mathbf{X}(t)) = (g_1(t,\mathbf{X}),\dots,g_P(t,\mathbf{X}))$$

is a stochastic process, whose $k-$th component is given by

$$\mathrm{d}Y^k = \frac{\partial g_k}{\partial t}(t,\mathbf{X})\,\mathrm{d}t + \sum_i \frac{\partial g_k}{\partial x_i}(t,\mathbf{X})\,\mathrm{d}X^i + \frac{1}{2}\sum_{i,j}\frac{\partial^2 g_k}{\partial x_i \partial x_j}(t,\mathbf{X})(\mathrm{d}X^i)(\mathrm{d}X^j), \tag{3}$$

where $\mathrm{d}X^i \cdot \mathrm{d}X^j$ is computed according to the rules $\mathrm{d}t \cdot \mathrm{d}t = \mathrm{d}t\cdot \mathrm{d}W^i = \mathrm{d}W^i \cdot \mathrm{d}t = 0$ and $\mathrm{d}W^i \cdot \mathrm{d}W^j = \delta_{i,j}\,\mathrm{d}t$.

2. SIMULATIONS OF THE STOCHASTIC SOLUTION

To simulate the solution of a stochastic differential equation numerical techniques have to be used (see [4]). The simplest numerical scheme, the stochastic Euler scheme, is based on numerical methods for ordinary differential equations.

Let us consider an equidistant discretisation of the time interval $\langle t_0,T\rangle$ as

$$t_n = t_0 + nh, \text{where} h = \frac{T-t_0}{n} = t_{n+1} - t_n = \int_{t_n}^{t_{n+1}} \mathrm{d}t$$

and the corresponding discretisation of the $j-$th component of the Wiener process as

$$\Delta W_n^j = W^j(t_{n+1}) - W^j(t_n) = \int_{t_n}^{t_{n+1}} \mathrm{d}W^j(s).$$

To be able to apply any stochastic numerical scheme, first we have to generate, for all j, the random increments of W^j as independent Gauss random variables with mean $E[\Delta W_n^j] = 0$ and $E[(\Delta W_n^j)^2] = h$.

The Euler scheme for the $i-$th component of an N dimensional stochastic differential equation has the form

$$X_{n+1}^i = X_n^i + A^i(t_n,\mathbf{X}_n)h + \sum_{j=1}^{M} B^{i,j}(t_n,\mathbf{X}_n)\Delta W_n^j. \tag{4}$$

For measuring the accuracy of a numerical solution to an SDE we use the strong order of convergence. We say that a stochastic numerical scheme converges with strong order γ if there exist real constants $K > 0$ and $\delta > 0$, so that

$$E[|X_T - X_T^h|] \leqq K h^\gamma, \quad h \in (0, \delta) \tag{5}$$

where the numerical solution is denoted by X_T^h. The Euler scheme converges with strong order $\gamma = \frac{1}{2}$.

3. MODELLING AN *RL* CIRCUIT WITH STOCHASTIC SOURCE

We want to apply the Itô stochastic calculus to an electrical problem like in [3]. We will consider inductor-resistor electrical circuits. The electrical current $i(t)$ at time t in a simple *RL* electrical circuit satisfies the differential equation

$$L\,\frac{\mathrm{d}\,i(t)}{\mathrm{d}t} + Ri(t) = v(t), \quad i(0) = i_0, \tag{6}$$

where the resistance R and the inductance L are constants and $v(t)$ denotes the potential source at time t (see [2]). Now let us allow some randomness in the potential source. Instead of $v(t)$ we consider the non deterministic version of this function:

$$v^*(t) = v(t) + \text{“noise”}.$$

To be able to substitute this into the equation of the circuit we have to describe mathematically the “noise”. It is reasonable to look at it as a stochastic process $\xi(t)$ called the “white noise process”. We get the following equation (α is a constant)

$$L\,\frac{\mathrm{d}\,i(t)}{\mathrm{d}t} + Ri(t) = v(t) + \alpha\xi(t), \quad i(0) = i_0. \tag{7}$$

From the mathematical point of view the white noise is not very suitable. We multiply the equation by $\mathrm{d}t$ and then we replace $\xi(t)\,\mathrm{d}t$ by a term $\mathrm{d}W(t)$. There $W(t)$ is the Wiener proces. Formally the “white noise” is the time derivative of the Wiener process $W(t)$. We get a stochastic differential equation

$$\mathrm{d}I(t) = \left(\frac{1}{L}v(t) - \frac{R}{L}I(t)\right)\mathrm{d}t + \frac{\alpha}{L}\,\mathrm{d}W(t), \tag{8}$$

where $I(0) = I_0$. We consider both the initial condition and the current at time t as random variables and denote them by capital letters.

To solve this equation we compute, using the Itô formula, the derivative of the function

$$\begin{aligned} &g(t, I(t)) = \mathrm{e}^{\frac{Rt}{L}} I(t): \\ &\mathrm{d}g(t, I(t)) = \mathrm{d}\left(\mathrm{e}^{\frac{Rt}{L}} I(t)\right) = \mathrm{e}^{\frac{Rt}{L}}\frac{R}{L}I(t)\;\mathrm{d}t + \mathrm{e}^{\frac{Rt}{L}}\;\mathrm{d}I(t) = \\ &= \mathrm{e}^{\frac{Rt}{L}}\left(\frac{R\,I(t)}{L}\,\mathrm{d}t + \frac{v(t)}{L}\,\mathrm{d}t - \frac{R\,I(t)}{L}\,\mathrm{d}t + \frac{\alpha}{L}\,\mathrm{d}W(t)\right) = \\ &= \mathrm{e}^{\frac{Rt}{L}}\frac{v(t)}{L}\,\mathrm{d}t + \mathrm{e}^{\frac{Rt}{L}}\frac{\alpha}{L}\,\mathrm{d}W(t). \end{aligned} \tag{9}$$

From this we get the solution

$$\begin{aligned} I(t) = \mathrm{e}^{-\frac{Rt}{L}}\, I(0) + \frac{1}{L}\int_0^t \mathrm{e}^{\frac{R(s-t)}{L}} v(s)\;\mathrm{d}s + \\ + \frac{\alpha}{L}\int_0^t \mathrm{e}^{\frac{R(s-t)}{L}}\;\mathrm{d}W(s). \end{aligned} \tag{10}$$

The solution $I(t)$ is a random process and for it’s expectation we have for every $t > 0$

$$E[I(t)] = \mathrm{e}^{\frac{-Rt}{L}} \cdot E[I_0] + \frac{1}{L}\int_0^t \mathrm{e}^{\frac{R(s-t)}{L}} \cdot v(s)\,\mathrm{d}s. \tag{11}$$

The second moment $D(t) = E[I(t)^2]$ can be computed as a solution of the ordinary differential equation

$$\frac{\mathrm{d}\,D(t)}{\mathrm{d}t} = \left(-\frac{2R}{L}\right) D(t) + 2m(t)\frac{v(t)}{L} + \frac{\alpha^2}{L^2}, \tag{12}$$

when $D(0) = E[I^2(0)]$ and $m(t) = E[I(t)]$.

The solution $I(t)$ is a Gaussian process. That means, that $I(t)$ is distributed $N(m(t), \sigma^2(t))$ for every $t \in \langle 0, T\rangle$, where $m(t) = E[I(t)]$ and $\sigma^2(t) = E[I(t)^2] - m^2(t)$. Based on the properties of the normal distribution, we can compute in any t, that

$$P(|I(t) - m(t)| < 1.96\;\sigma(t)) = 2\;\Phi(1.96) - 1 = 0.95,$$

where

$$\Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}s^2}\;\mathrm{d}s. \tag{13}$$

As we are able to compute $E[I(t)^2]$ and $m(t) = E[I(t)]$, we can predict with a probability 95% the interval $(m(t) - \varepsilon,\; m(t) + \varepsilon)$, where the trajectories of the stochastic solution take place.

Example 1. Let us consider the *RL* electrical circuit, when L, R and $v(t)$ are constants, $I(0) = 0$, $\alpha = 1$. In that case the expectation of the stochastic solution is equal to the classical solution of the circuit. Using the Euler method we plot the stochastic solution together with the deterministic solution (Fig. 1).

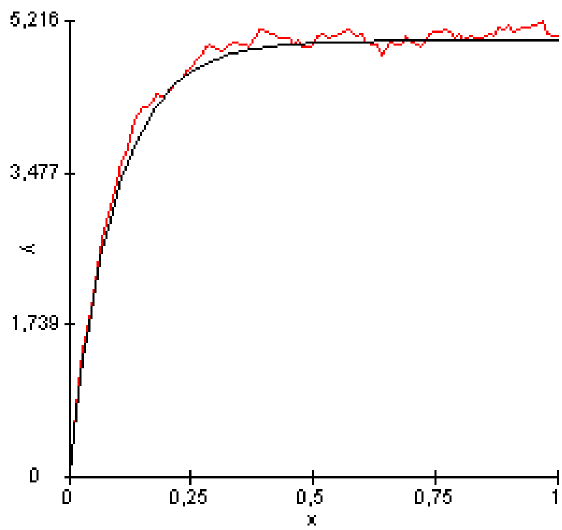

Fig. 1 The deterministic solution and a sample path of the stochastic solution

In Fig. 2 we graph the 95% prediction interval for the trajectories of the stochastic solution. This result was verified in an experiment by measurements on inductor-resistor electrical circuits.

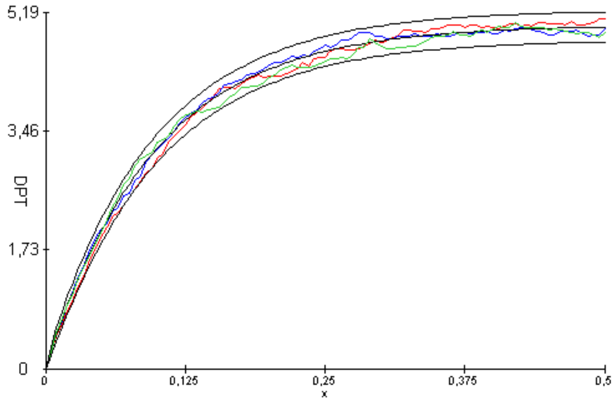

Fig. 2 Three trajectories of the stochastic solution and a 95% prediction interval

4. *RL* CIRCUITS WITH TWO STOCHASTIC PARAMETERS

Now we allow some randomness in the electrical source as well as in the resistance. We consider

$$v^*(t) = v(t) + \text{“noise”}, \qquad R^* = R + \text{“noise”}.$$

The stochastic differential equation describing this situation is

$$\mathrm{d}I(t) = \frac{v(t) - RI(t)}{L}\,\mathrm{d}t - \frac{\beta}{L} I(t)\,\mathrm{d}W_2(t) + \frac{\alpha}{L}\,\mathrm{d}W_1(t),$$

$I(0) = I_0$, where α and β are non negative constants. Their magnitudes determine the deviation of the stochastic case from the deterministic one. To find the solution, we first have to define a function $F(t) = \mathrm{e}^{\frac{R}{L}t + \frac{\beta^2}{2L^2}t + \frac{\beta}{L}W_2(t)}$, (see [5], p. 77, exercise 5.16) and then compute

$$\mathrm{d}F(t)\,I(t) = \mathrm{d}\left(\mathrm{e}^{\frac{R}{L}t + \frac{\beta^2}{2L^2}t + \frac{\beta}{L}W_2(t)}\, I(t)\right). \tag{14}$$

applying the multidimensional Itô formula to the function $g(t,x,y) : \langle 0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$,

$$g(t,x,y) := \mathrm{e}^{\frac{R}{L}t + \frac{\beta^2}{2L^2}t + \frac{\beta}{L}x} y. \tag{15}$$

We obtain

$$\begin{aligned}
&\mathrm{d}g(t, W_2(t), I(t)) = \mathrm{d}\left(\mathrm{e}^{\frac{R}{L}t + \frac{\beta^2}{2L^2}t + \frac{\beta}{L}W_2(t)} I(t)\right) = \\
&= \mathrm{d}\,(F(t)I(t)) = F(t)\left(\frac{R}{L} + \frac{\beta^2}{2L^2}\right) I(t)\,\mathrm{d}t + \\
&+ F(t)\frac{\beta}{L} I(t)\,\mathrm{d}W_2(t) + F(t)\,\mathrm{d}I(t) + \\
&+ \frac{1}{2}\frac{\beta^2}{L^2} F(t)I(t)\underbrace{(\mathrm{d}W_2(t))^2}_{=\mathrm{d}t} + F(t)\frac{\beta}{L}(\mathrm{d}W_2(t)\,\mathrm{d}I(t)) = \\
&= F(t)\left(\left(\frac{R}{L} + \frac{\beta^2}{2L^2}\right) I(t)\,\mathrm{d}t + \frac{\beta}{L} I(t)\,\mathrm{d}W_2(t)\right) + \\
&+ F(t)\left(\frac{v(t) - RI(t)}{L}\,\mathrm{d}t - \frac{\beta}{L} I(t)\,\mathrm{d}W_2(t) + \frac{\alpha}{L}\,\mathrm{d}W_1(t)\right) + \\
&+ F(t)\left(\frac{1}{2}\frac{\beta^2}{L^2} I(t)\,\mathrm{d}t + \frac{\beta}{L}\left(-\frac{\beta}{L} I(t)\right)\mathrm{d}t\right) = \\
&= F(t)\left(\frac{v(t)}{L}\,\mathrm{d}t + \frac{\alpha}{L}\,\mathrm{d}W_1(t)\right).
\end{aligned} \tag{16}$$

Now there is no $I(t)$ on the righthand side of the equation. We have

$$\mathrm{d}\,(F(t)\,I(t)) = F(t)\left(\frac{v(t)}{L}\,\mathrm{d}t + \frac{\alpha}{L}\,\mathrm{d}W_1(t)\right). \tag{17}$$

We can write:

$$\begin{aligned}
&F(t)I(t) - F(0)I(0) = \\
&= \frac{1}{L}\int_0^t v(s)F(s)\,\mathrm{d}s + \frac{\alpha}{L}\int_0^t F(s)\,\mathrm{d}W_1(t).
\end{aligned} \tag{18}$$

Multiplying this equation by $F^{-1}(t)$ we get the stochastic solution of the *RL* circuit as a stochastic process

$$\begin{aligned}
I(t) &= I_0 \mathrm{e}^{-\frac{R}{L}t - \frac{\beta^2}{2L^2}t - \frac{\beta}{L}W_2(t)} + \\
&+ \frac{1}{L}\int_0^t v(s)\mathrm{e}^{\frac{R}{L}(s-t) + \frac{\beta^2}{2L^2}(s-t) + \frac{\beta}{L}(W_2(s) - W_2(t))}\,\mathrm{d}s + \\
&+ \frac{\alpha}{L}\int_0^t \mathrm{e}^{\frac{R}{L}(s-t) + \frac{\beta^2}{2L^2}(s-t) + \frac{\beta}{L}(W_2(s) - W_2(t))}\,\mathrm{d}W_1(s).
\end{aligned} \tag{19}$$

Reference [1], p. 142–143, shows that if $E[I_0^2] < \infty$, the expectation $E[I(t)] = m(t)$ is the solution of the ordinary differential equation

$$\frac{\mathrm{d}\,m(t)}{\mathrm{d}t} = \frac{1}{L}(v(t) - Rm(t)), \quad m(0) = E[I_0]. \tag{20}$$

We can easily compute that

$$E[I(t)] = \mathrm{e}^{\frac{-Rt}{L}} \cdot E[I_0] + \frac{1}{L}\int_0^t \mathrm{e}^{\frac{R(s-t)}{L}} \cdot v(s)\,\mathrm{d}s, \tag{21}$$

for every $t > 0$. If the random variable $I(0) = I_0$ is constant, then the expectation of the stochastic solution is equal to the deterministic solution of the circuit. So the function $m(t) = E[I(t)]$ is independent of the fluctuational part of the stochastic differential equation.

The second moment $D(t) = E[I^2(t)]$ is the solution of the ordinary linear equation

$$\frac{\mathrm{d}\,D(t)}{\mathrm{d}t} = \left(\frac{\beta^2}{L^2} - \frac{2R}{L}\right) D(t) + 2m(t)\frac{v(t)}{L} + \frac{\alpha^2}{L^2}, \tag{22}$$

$D_0 = E[I_0^2]$, where $m(t) = E[I(t)]$.

In this case $I(t)$ is not a Gaussian process, but we can get some prediction interval $(m(t) - \varepsilon,\ m(t) + \varepsilon)$ for the trajectories of the stochastic solution using the Chebyshev's inequality. For $t \in \langle 0, T\rangle$

$$P(|I(t) - m(t)| < 2\,\sigma(t)) \geqq 0.75 \tag{23}$$

or

$$P(|I(t)-m(t)|<3\,\sigma(t)) \geqq \frac{8}{9} = 0.\overline{8}. \qquad (24)$$

Example 2. Let us again consider the RL electrical circuit, when L, R and $v(t)$ are constants, $I_0 = 0$, $\alpha = 1$, $\beta = 1$. Using the Euler scheme we compute and graph some sample paths of the stochastic solution of the RL circuit together with a prediction interval computed from the Chebyshev's inequality.

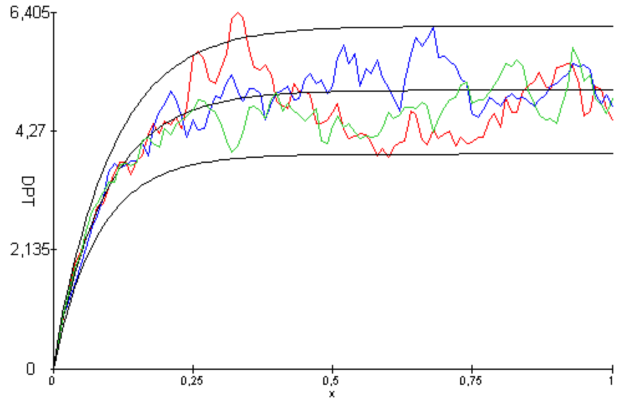

Fig. 3 Three trajectories of the stochastic solution and a prediction interval, where $P(|I(t)-m(t)|<2\,\sigma(t)) \geqq 0.75$

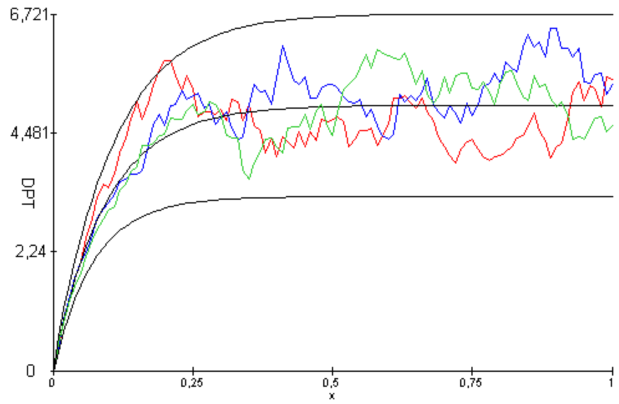

Fig. 4 Three trajectories of the stochastic solution and a prediction interval, where $P(|I(t)-m(t)|<3\,\sigma(t)) \geqq 0.\overline{8}$

The pictures are generated in C# (see [7] and [6]).

5. CONCLUSION

This paper shows some applications of the Itô stochastic calculus to the problems of modelling inductor-resistor electrical circuits. The deterministic model of the circuit is replaced by a stochastic model by adding a noise term to the source, in second case in both the source and the resistance. The analytic solution of the resulting stochastic differential equations are obtained using the multidimensional Itô formula. Statistical estimates of the stochastic solutions are examined and confidence intervals are found for the trajectories of the solution. The programming language C#, a part of the new MS .NET platform, is used for numerical simulations. The stochastization of the problem contains two independent white noises, which are artificial, but the results were verified in an experiment by measurements on inductor-resistor electrical circuits.

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BIOGRAPHY

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