ON THE NEURODYNAMIC MODEL OF RECURRENT NETWORKS

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SUMMARY

Neurodynamical models of recurrent networks are characterized by their topology, by interactions between the elements sitting at the nodes of a network and by intrinsic dynamics of these local subsystems. In the present paper we give some relationship between Lyapunov's exponents and the recurrent neural network model described by the system of delay-differential equations. We investigate the dynamic properties of the specific class of nonlinear delay-differential equations by studying the asymptotic behaviour of their solutions by means of Lyapunov's exponents.

Keywords: *neural networks, Lyapunov's exponents, Cauchy's matrix, neurodynamical systems, delay–differential equations*

1. INTRODUCTION

New technologies in engineering and biomedicine are creating problems in which nonstationarity, nonlinearity and complexity play a major role. Solutions to many of these problems require the use of nonlinear processors, among which neural networks are one of the most powerful. Neural networks are appealing because they learn by example and are strogly supported by statistical and optimisation theories. The area of neural networks is nowadays considered from two main perspectives. The first perspective is cognitive science, which is an interdisciplinary study of the mind.The second perspective is connectionism, which is a theory of information processing. The neural networks model in this paper is approached from an engineering perspective, i.e. to make networks efficient in terms of topology and capture dynamics of time–varying systems. Neural dynamics in that case can be considered from two aspects, convergence of state variables (memory recall) and the number, position, local stability and domains of attraction of equilibrium states (memory capacity). When we speak of stability in the context of a nonlinear dynamical system, we ussually mean stability in the sense of Lyapunov. A. M. Lyapunov (see [8]) presented the fundamental concepts of the stability theory known as the first method of Lyapunov. This method is widely used for the stability analysis of linear and nonlinear systems, both time-invariant and time-varying. As such it is directly aplicable to the stability analysis of neural networks. The study of neurodynamics may follow one of two routes, depending on the application of interest:

1. Deterministic neurodynamics, in which the neural network model has a deterministic behaviour. In mathematical terms, it is described by a set of nonlinear delay-differential equations that define the exact evolution of the model as a function of time.

2. Statistical neurodynamics, in which the neural network model is perturbed by the presence of noise. In this case, we have to deal with stochastic nonlinear differential equations, expressing the solution in probabilistic terms. The combination of stochasticity and nonlinearity makes the subject more difficult to handle.

In this paper we restrict ourselves to deterministic neurodynamics.

2. LYAPUNOV'S EXPONENTS

In order to proceed with the study of neurodynamics, we need a mathematical model for describing the dynamics of a nonlinear system. A model most naturally suited for this purpose is the so-called state-space model. According to this model, we think in terms of a set of state variables whose values are supposed to contain sufficient information to predict the future evolution of the system. Let

$$x_1(t), x_2(t), \dots, x_n(t) \tag{1}$$

denote the state variables of a nonlinear dynamical system, where continuous time t is the independent variable and n is the order of the system. The dynamics of a large class of nonlinear dynamical systems may then be cast in the form of a system of first-order differential equations written as follows:

$$\frac{d}{dt}x_i(t) = p_i(t)x_i(t) + \sum_{j=1}^{n} q_{ij}(t)u_j(t) + \\ + \sum_{j=1}^{n} d_{ij}(t)y_j(t) + I_i(t), \quad i = 1,2,\dots,n, \tag{2}$$

where all functions

$$p_i(t),\ q_{ij}(t),\ d_{ij}(t),\ I_i(t) \tag{3}$$

are assumed to be continuous functions of time,

$$p_i(t) < 0, \quad |p_i(t)| \le p < +\infty,$$

$$\left|q_{ij}(t)\right| \le q < +\infty, \quad \left|d_{ij}(t)\right| \le d < +\infty, \tag{4}$$

$$p \ge \sqrt{\sum_{i=1}^{n} p_i^2(t)}, \quad q \ge \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} q_{ij}^2(t)},$$

$$d \ge \sqrt{\sum_{j=1}^{n}\sum_{i=1}^{n} d_{ij}^2(t)},$$

$$u_j(t) = \frac{1}{2} y_j^2(t) \left(\left| r_j(t) + 1 \right| - \left| r_j(t) - 1 \right| \right), \tag{5}$$

$$r_j(t) = \sqrt{\sum_{i=1}^{n} x_i^2(t-\tau) - x_j^2(t-\tau)}, \ \ \tau > 0, \ \ t - \tau < t_0,$$

$$y_j(t) = \frac{1}{2} \left(\left| x_j(t) + 1 \right| - \left| x_j(t) - 1 \right| \right), \tag{6}$$

$i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n.$

This system of delay-differential equations can be used to model recurrent neural networks.

The initial value problem for (2) is defined as follows:
On the initial set

$$E_{t_0} = \{ t - \tau : t - \tau < t_0, \, t \in \langle t_0, \infty) \} \cup \{ t_0 \} \tag{7}$$

let a continuous initial vector functions

$$\varphi(t) = (\varphi_0(t), \varphi_1(t), \varphi_2(t), \ldots \varphi_{n-1}(t)) \tag{8}$$

be given.
We have to find the solution

$$x(t) = (x_1(t), x_2(t), \ldots, x_n(t)), \quad x_i(t) \in C^1 (\langle t_0, \infty)) \tag{9}$$

of (2) satisfying

$$x_{j+1}(t) = \varphi_j(t), \quad j = 0, 1, 2, \ldots, n-1, \tag{10}$$

if

$$t - \tau \le t \le t_0, \quad i = 1, 2, \ldots, n. \tag{11}$$

Under the above assumptions, the initial value problem (2),(10) has exactly one solution on the interval $\langle t_0, \infty)$, where

$$\varphi_j(t) = x_{j+1,0} \psi_j(t), \quad x_{j+1}(t_0) = x_{j+1,0}, \quad \psi_j(t_0) = 1, \quad j = 0, 1, \ldots, n-1. \tag{12}$$

In the following we consider the system of linear differential equations of the form

$$\frac{d}{dt} x_i(t) = p_i(t) x_i(t), \quad i = 1, 2, \ldots, n \tag{13}$$

and the system of nonlinear delay-differential equations of the form

$$\frac{d}{dt} x_i(t) = p_i(t) x_i(t) + \sum_{j=1}^{n} q_{ij}(t) u_j(t) + \sum_{j=1}^{n} d_{ij}(t) y_j(t), \quad i = 1, 2, \ldots, n. \tag{14}$$

Definition 1.1. A superior Lyapunov's exponent of a vector function $x(t)$ is called a real number $\overline{\lambda}$ which is defined by

$$\overline{\lambda} = \limsup_{t \to \infty} \left(\frac{1}{t} \ln \| x(t) \| \right).$$

Definition 1.2. A inferior Lyapunov's exponent of a vector function $x(t)$ is called a real number $\underline{\lambda}$ which is defined by

$$\underline{\lambda} = \liminf_{t \to \infty} \left(\frac{1}{t} \ln \| x(t) \| \right),$$

where

$$\| x \| = \sqrt{(x,x)}, \quad (x,y) = \sum_{i=1}^{n} x_i \overline{y_i}.$$

Definition 1.3. A superior central exponent of a Cauchy's matrix of a linear differential system is called a real number Ω which is defined by

$$\Omega = \inf_{T>0} \left(\limsup_{k \to \infty} \frac{1}{kT} \sum_{i=1}^{k} \ln \| Z(iT) \| \right) = \lim_{T \to \infty} \left(\limsup_{k \to \infty} \frac{1}{kT} \sum_{i=1}^{k} \ln \| Z(iT) \| \right), \tag{15}$$

where

$$Z(iT) = X(iT, (i-1)T).$$

Definition 1.4. A inferior central exponent of a Cauchy's matrix of a linear differential system is called a real number ω which is defined by

$$\omega = \inf_{T>0} \left(\limsup_{k \to \infty} \frac{1}{kT} \sum_{i=1}^{k} \ln \| Y(iT) \|^{-1} \right) = \lim_{T \to \infty} \left(\limsup_{k \to \infty} \frac{1}{kT} \sum_{i=1}^{k} \ln \| Y(iT) \|^{-1} \right), \tag{16}$$

where

$$Y(iT) = X^{-1}(iT, (i-1)T).$$

We have to find the norm of a Cauchy's matrix of the linear differential system by using the following formula

$$\| X(t,s) \| = \max_x \frac{\| x(t) \|}{\| x(s) \|}, \tag{17}$$

where we have to search a maximum element of a set of al solutions of a linear differential system.

Choose any nontrivial solution

$$w(t) = (w_1(t), w_2(t), \ldots, w_n(t)) \tag{18}$$

of the set of all solutions of (14). If $a_{wij}(t)$ denotes

$$q_{ij}(t) \cdot h_j(t), \quad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, n$$

and

$$h_j(t) = \frac{1}{2} \left(\left| v_j(t) + 1 \right| - \left| v_j(t) - 1 \right| \right),$$

$$v_j(t) = \sqrt{\sum_{i=1}^{n} w_i^2(t-\tau) - w_j^2(t-\tau)}, \quad \tau > 0, \quad t - \tau < t_0, \tag{19}$$

then $w(t)$ is the solution of the nonlinear differential system

$$\frac{d}{dt}z_i(t) = p_i(t)z_i(t) + \sum_{j=1}^{n} a_{wij}(t)y_j^2(t) + \\ + \sum_{j=1}^{n} d_{ij}(t)y_j(t), \quad i = 1,2,\dots,n \tag{20}$$

too. The equality

$$a_{wij}(t) = q_{ij}(t) \cdot h_j(t) \tag{21}$$

implies the fact that all coefficiens $a_{wij}(t)$ are continuous functions of time and

$$\left|h_j(t)\right| \le 1, \quad \left|a_{wij}(t)\right| \le \left|q_{ij}(t)\right| \cdot \left|h_j(t)\right| \le q, \tag{22}$$
$$t \in \langle 0, +\infty), \quad 0 < q < +\infty.$$

Lemma 1.1. Let $f(t)$ be a continuous function of time and

$$g(t) = \frac{1}{2}(|f(t)+1| - |f(t)-1|).$$

Then,

$$g^4(t) \le g^2(t) \le f^2(t). \tag{23}$$

Proof: Suppose that $f(t) \ge 1$. Then,

$$g(t) = \frac{1}{2}(|f(t)+1| - |f(t)-1|) = \\ = \frac{1}{2}(f(t)+1-(f(t)-1)) = 1$$

and

$$g^4(t) \le g^2(t) = 1 \le f^2(t). \tag{24}$$

Suppose that $-1 \le f(t) \le 1$. Then,

$$g(t) = \frac{1}{2}(|f(t)+1| - |f(t)-1|) = \\ = \frac{1}{2}(f(t)+1-(-f(t)+1)) = f(t)$$

and

$$g^4(t) \le g^2(t) = f^2(t). \tag{25}$$

Suppose that $f(t) \le -1$. Then,

$$g(t) = \frac{1}{2}(|f(t)+1| - |f(t)-1|) = \\ = \frac{1}{2}(-(f(t)+1)-(-(f(t)-1))) = -1$$

and

$$g^4(t) = g^2(t) = 1 \le f^2(t). \tag{26}$$

Consequently,

$$g^4(t) \le g^2(t) \le f^2(t), \quad t \in (-\infty, \infty) \tag{27}$$

and this proves Lemma 1.1.

Corollary 1.1. Let $f_i(t)$ be a continuous function of time and

$$g_i(t) = \frac{1}{2}(|f_i(t)+1| - |f_i(t)-1|), \quad i = 1,2,\dots,n.$$

Then,

$$\|r(g(t))\| \le \|g(t)\| \le \|f(t)\|, \tag{28}$$

where

$$f(t) = (f_1(t), f_2(t), \dots, f_n(t)),$$
$$g(t) = (g_1(t), g_2(t), \dots, g_n(t)),$$
$$r(g(t)) = (g_1^2(t), g_2^2(t), \dots, g_n^2(t)),$$
$$\|f(t)\| = \sqrt{\sum_{i=1}^{n} f_i^2(t)}, \quad \|g(t)\| = \sqrt{\sum_{i=1}^{n} g_i^2(t)},$$
$$\|r(g(t))\| = \sqrt{\sum_{i=1}^{n} g_i^4(t)}.$$

Theorem 1.1. Let $q \in R$ satisfies the inequality (22). Then, every nontrivial solution $z(t)$ of nonlinear differential system (14) satisfies the inequality

$$e^{-M(t-t_0)} \le \frac{\|z(t)\|}{\|z(t_0)\|} \le e^{M(t-t_0)}, \\ t \ge t_0, \quad M = p+q+d. \tag{29}$$

Proof: Due to the fact that all constants q do not depend on the parameter w, there suffices to prove this theorem for all nontrivial solutions of (20).

In the first part of the proof we show that any nontrivial solution $z(t)$ of (20) satisfies the inequality

$$\left|\frac{d}{dt}\|z(t)\|^2\right| \le 2M\|z(t)\|^2. \tag{30}$$

Make the modify of the left hand side of (30) gives

$$\begin{aligned}\left|\frac{d}{dt}\|z(t)\|^2\right| &= \left|\frac{d}{dt}(z(t),z(t))\right| = \\ &= \left|\big(z'(t),z(t)\big) + \big(z(t),z'(t)\big)\right| = \\ &= 2\left|\big(z'(t),z(t)\big)\right| \le 2\left\|z'(t)\right\| \cdot \|z(t)\| = \\ &= 2 \cdot \sqrt{\sum_{i=1}^{n}(z_i'(t))^2} \cdot \|z(t)\| = \\ &= 2 \cdot \Big(\sum_{i=1}^{n}(p_i(t)z_i(t) + a_{wi1}(t)y_1^2(t) + \dots \\ &\dots + a_{win}(t)y_n^2(t) + d_{i1}(t)y_1(t) + \dots \\ &\dots + d_{in}(t)y_n(t))^2\Big)^{\frac{1}{2}} \cdot \|z(t)\| \le \\ &\le 2 \cdot \Big(\sqrt{p_1^2(t)z_1^2(t)} + \sqrt{p_2^2(t)z_2^2(t)} + \dots \\ &\dots + \sqrt{p_n^2(t)z_n^2(t)} + \sqrt{\sum_{i=1}^{n} a_{wi1}^2(t)y_1^4(t)} + \dots \\ &\dots + \sqrt{\sum_{i=1}^{n} a_{win}^2(t)y_n^4(t)} + \sqrt{\sum_{i=1}^{n} d_{i1}^2(t)y_1^2(t)} + \dots \\ &\dots + \sqrt{\sum_{i=1}^{n} d_{in}^2(t)y_n^2(t)}\Big) \cdot \|z(t)\| \le\end{aligned}$$

$$\leq 2 \cdot (\sqrt{\sum_{j=1}^{n} p_j^2(t)} \cdot \|z(t)\| +$$
$$+ \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} a_{wij}^2(t)} \cdot \|r(y(t))\| +$$
$$+ \sqrt{\sum_{j=1}^{n} \sum_{i=1}^{n} d_{ij}^2(t)} \cdot \|y(t)\|) \cdot \|z(t)\| .$$

Returning to the inequality (28), we see that

$$\|r(y(t))\| \leq \|y(t)\| , \quad \|r(y(t))\| = \sqrt{\sum_{i=1}^{n} y_i^4(t)}.$$

Hence,

$$\left| \frac{d}{dt} \|z(t)\|^2 \right| \leq 2p \|z(t)\|^2 +$$
$$+2q \|r(y(t))\| \cdot \|z(t)\| + 2d \|y(t)\| \cdot \|z(t)\| \leq$$
$$\leq 2p \|z(t)\|^2 + (2q + 2d) \|y(t)\| \cdot \|z(t)\| \leq$$
$$\leq 2p \|z(t)\|^2 + (2q + 2d) \|z(t)\|^2 =$$
$$= 2 \cdot (a + q + d) \cdot \|z(t)\|^2 = 2M \|z(t)\|^2 . \tag{31}$$

The first part of the proof is complete.

In the second part of the proof multiplying both sides of this inequality by $\|z(t)\|^{-2}$, one may obtain

$$-M \leq \frac{d}{dt} \ln \|z(t)\| \leq M. \tag{32}$$

Integration of (32) gives

$$-M(t - t_0) \leq \ln \frac{\|z(t)\|}{\|z(t_0)\|} \leq M(t - t_0). \tag{33}$$

Consequently,

$$e^{-M(t-t_0)} \leq \frac{\|z(t)\|}{\|z(t_0)\|} \leq e^{M(t-t_0)}.$$

Notice that the solution $w(t)$ satisfies the inequality (29), too.
The proof is complete.

Remark: Implicit in this theorem is the fact, that if $z(t)$ satisfies the inequality (29) then Lyapunov's exponents satisfy the inequality

$$-M \leq \underline{\lambda} \leq \overline{\lambda} \leq M, \quad -M \leq \omega \leq \Omega \leq M. \tag{34}$$

3. CONCLUSION

The discovery that deterministic dynamical systems can have a very complicated (chaotic) behaviour brought about the notions of space–time chaos, coherent structures, intermittency, etc. Nonlinear dynamical systems of order greater than 2 have the capability of exhibiting a chaotic behaviour that is higly complex. Lyapunov's exponents can be used to study a chaotic behaviour of solutions of neurodynamical systems, too.

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