MONADIC PREDICATE FORMULAE

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SUMMARY

It may be easily to fix truthfulness value of monadic predicate formulae on the arbitrary cardinality subject domain. Because is often tested the correctness of statements reasoning, theirs premises and conclusion can be expressed through in this paper described formulae, it is very suitable to occupy with the monadic predicate formulae.

Keywords : *logic formulae, predicate formulae, truths of formulae, partition, Karnaugh map*

1. INTRODUCTION

The truthfulness value of arbitrary monadic predicate formulae is possible always and simple to determine, i.e. predicate formulae, in which occur only one-placed of predicate independent on the cardinality of subject domain. It is consequently suited with described formulae of predicate logic of the first order to occupy one self.

2. DESCRIBING OF MONADIC PREDICATE FORMULAE

Assume a non-empty subject domain D *($D \neq \emptyset$)* of arbitrary cardinality (a finite, countable also an uncountable one). Let an individual **unary relation** r_i $(r_i \subseteq D)$ be given on the domain D and a one-place **individual predicate** p_i ,i.e. $p_i : D \to \{0,1\}$: $x \mapsto 0$ for $x \notin r_i$ and $x \mapsto 1$ for $x \in r_i$, carried by the relationship r_i . Each unary predicate $p_i\ (x)$ defines on the subject domain $D(x \in D)$ a dichotomy (a partition having two classes D_i^0, D_i^1) $\pi_{p_i}\ (D) = \{D_i^\sigma\}_{\sigma \in \{0,1\}}$ such that $p_i\,(x) = \begin{cases} 0 \text{ for } x \in D_i^0 \\ 1 \text{ for } x \in D_i^1 \end{cases}$ and vice versa. Note that the nullary predicate $\{\mathrm{d}\} \to \{0,1\}$ (d $\in D$) defines on D a monadic partition *$\pi(D) = \{D\}$*.

Be given a system $\{p_i\,(x)\}_{i=1}^m$ of unary predicates $p_i\,(x)$ on D. The system of predicates defines on D a generally incomplete partition

$$\pi_{p_1,p_2,\ldots,p_m}(D) = \left\{\bigcap_{i=1}^{m} D_i^{\sigma_i}\right\}_{\langle\sigma_1,\sigma_2,\ldots,\sigma_m\rangle \in \{0,1\}^m}$$

having $\bigcap_{i=1}^{m} D_i^{\sigma_i}$ $\left|\,\pi_{p_1,p_2,\ldots,p_m}(D)\,\right| \left(\leq 2^m\right)$ classes of $\bigcap_{i=1}^{m} D_i^{\sigma_i}$, and on each class $\bigcap_{i=1}^{m} D_i^{\sigma_i}$, or shortened to the predicates $p_i\,(x)$ assume a value σ_i too. The **partition** $\pi_{p_1,p_2,\ldots,p_m}(D)$ is **incomplete** (**complete**) if the union of all classes of it is a proper subset (is equal to) D.

Example 1.: A system, $\{p_i\ (x)\ \}_{i=1}^3$ of predicates *$p_i(x)$* can determine the partition $\pi_{p_1,p_2,p_3}(D) = \left\{D_1^0 D_2^0 D_3^1,\ D_1^1 D_2^0 D_3^1,\ D_1^1 D_2^1 D_3^0,\ D_1^1 D_2^1 D_3^1\ \right\}$ on D. The partition is incomplete because of $\left|\pi_{p_1,p_2,p_3}(D)\ \right| = 4\ \left(\langle\ 2^3\right)$; on a class, e.g., $D_1^1 D_2^1 D_3^0$ the predicates assume the values $p_1\,(x) = p_2\,(x) = 1$ a $p_3\,(x) = 0$.

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If the subject domain D in a plane is depicted by a rectangle or square, the incomplete as well as the complete partitions on D defined by the system of predicates on D can be conveniently depicted by means of k-dimensional ($k = 0,1,\ldots,m$) **Karnaugh map** M_k which is recurrently defined for systems consisting of a nullary predicate - M_o, a single $p_i(x) - M_1$ predicate, and m predicates $p_i(x) - M_m$, as can be seen in Fig. 1 (freely according to [1]).

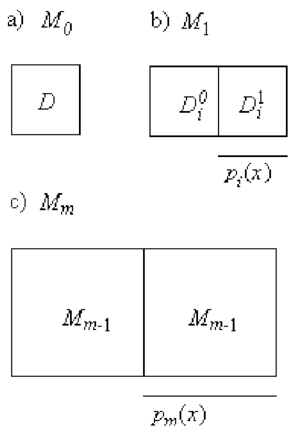

Fig. 1 Recurrent definition of Karnaugh map M_k

Example 2.: Depict the incomplete partition D from Example 1 using a Karnaugh map M_3 – Fig. 2.

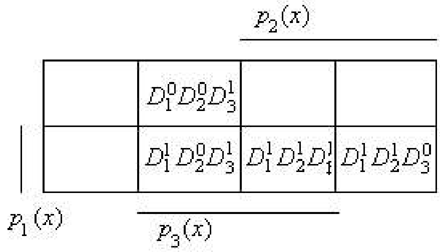

Fig. 2 Incomplete partition D from Example 2

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Let the formula of the first-order predicate logic comprising only unary predicates over the domain D be called a ***monadic predicate formula***, e.g., $\forall x\, p(x) \Rightarrow \exists x\, p(x)$. Each monadic predicate formula over the domain D comprising m different unary predicates $p_i(x)$ ($i = 1,2,...,m$) is apparently satisfiable in D only and only if it is satisfiable in the classes of the incomplete partition $\pi_{p_1,p_2,...,p_m}(D)$ [2]. On the other hand, it is a tautology if and only if it is satifiable on all classes of the complete partition $\pi_{p_1,p_2,...,p_m}(D)$.

If we denote $\pi_{p_1,p_2,...,p_m}(D) = \{\delta_j\}_{j=1}^{n}$, where $n \leq 2^m$, then $\forall x\, p_i(x) = \mathop{\&}\limits_{j=1}^{n} p_i(\delta_j)$ and $\exists x\, p_i(x) = \bigvee\limits_{j=1}^{n} p_i(\delta_j)$. In the Karnaugh map we will mark $\forall x\, p_i(x)$ by hatching all squares of $p_i(d)$ and $\exists x\, p_i(x)$ by griddling all squares of $p_i(d)$ [3]. Instead of inconvenient griddling we can place a single dot in the particular square of the map.

The monadic predicate formula P appears to be a tautology if its general closure is a tautology, e.g., if $\forall u_1 \forall u_2 ... \forall u_l P$ is a tautology, in case that u_k (k = = 1,2,...,l) are all different free variables in the formula P. On the other hand the formula P is a tautology if the each square of its Karnaugh map is griddling or includes a dot too.

Example 3.: Determine both by formally and mapping procedures if the formulae below are tautologies ($\Rightarrow$ is the implication operator):

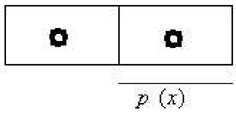
a) $\forall x\, p(x) \Rightarrow \exists x\, p(x)$,

b) $p(x)q(x) \Rightarrow (\exists x\, p(x))q(x)$,

c) $p(x)q(x) \Rightarrow (\forall x\, p(x))q(x)$

ad a) on the one hand

$$\models \forall x\, p(x) \Rightarrow \exists x\, p(x) = \overline{\forall x}\, p(x) \vee \exists x\, p(x) =$$
$$= \exists x\left(\overline{p(x)} \vee p(x)\right)$$

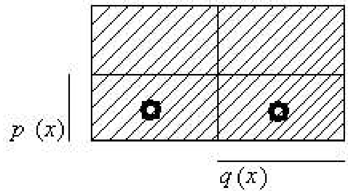


where |= means „*to be a tautology*“, on the other hand because $\pi_p(D) = \{\delta_1, \delta_2\}$, we obtain

$$\models p(\delta_1)\, p(\delta_2) \Rightarrow p(\delta_1) \vee p(\delta_2) =$$
$$= \overline{p(\delta_1)} \vee \overline{p(\delta_2)} \vee p(\delta_1) \vee p(\delta_2),$$

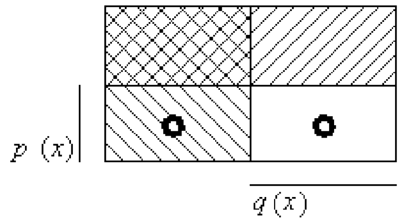
ad b) a general closure of the given formula is

$$\forall x\, \left(p(x)q(x) \Rightarrow (\exists x\, p(x))\, q(x)\right) =$$
$$= \forall x\left(\overline{p(x)} \vee \overline{q(x)} \vee (\exists x\, p(x))\, q(x)\right) =$$
$$= \forall x\left(\overline{p(x)} \vee \overline{q(x)} \vee \exists x\, p(x)\right)$$



and because $\pi_{p,q}(D) = \{\delta_j\}_{j=1}^{4}$, we obtain

$$\models\left(\overline{p(\delta_1)} \vee \overline{q(\delta_1)} \vee p(\delta_1) \vee p(\delta_2) \vee p(\delta_3) \vee p(\delta_4)\right)\&$$
$$\&\left(\overline{p(\delta_2)} \vee \overline{q(\delta_2)} \vee p(\delta_1) \vee p(\delta_2) \vee p(\delta_3) \vee p(\delta_4)\right)\&$$
$$\&\left(\overline{p(\delta_3)} \vee \overline{q(\delta_3)} \vee p(\delta_1) \vee p(\delta_2) \vee p(\delta_3) \vee p(\delta_4)\right)\&$$
$$\&\left(\overline{p(\delta_4)} \vee \overline{q(\delta_4)} \vee p(\delta_1) \vee p(\delta_2) \vee p(\delta_3) \vee p(\delta_4)\right)$$



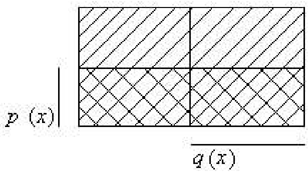
ad c) a general closure of the given formula is

$$\forall x\left(p(x)q(x) \Rightarrow \forall x\, p(x)q(x)\right) =$$
$$= \forall x\left(\overline{p(x)} \vee \overline{q(x)} \vee \forall x\, p(x)q(x)\right) =$$
$$= \forall x\left(\overline{p(x)} \vee \overline{q(x)} \vee \forall x\, p(x)\right)$$

and because $\pi_{pq}(D) = \{\delta_j\}_{j=1}^{4}$ we obtain

$$\left(\overline{p(\delta_1)} \vee \overline{q(\delta_1)} \vee p(\delta_1)\, p(\delta_2)\, p(\delta_3)\, p(\delta_4)\right)\&$$
$$\&\left(\overline{p(\delta_2)} \vee \overline{q(\delta_2)} \vee p(\delta_1)\, p(\delta_2)\, p(\delta_3)\, p(\delta_4)\right)\&$$
$$\&\left(\overline{p(\delta_3)} \vee \overline{q(\delta_3)} \vee p(\delta_1)\, p(\delta_2)\, p(\delta_3)\, p(\delta_4)\right)\&$$
$$\&\left(\overline{\mathrm{p}(\delta_4)} \vee \overline{q(\delta_4)} \vee p(\delta_1)\, p(\delta_2)\, p(\delta_3)\, p(\delta_4)\right)$$

which is not a tautology, and more over, because of the way in which the formula is organized it is not very convincing:



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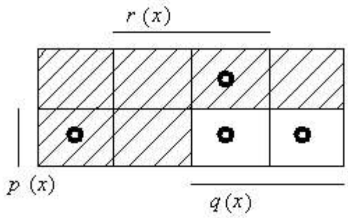
Karnaugh maps can be conveniently applied for examining correctness of statements reasoning, in case the premises and conclusions are monadic predicate formulae over *D,* are since if Horn's reasoning $p_1, p_2,...,p_m \models q$ *is given,* where p_i *(i* =1,2,...,*m*) premises, *q* is the conclusion, and $\models$ is the sign of inferring, then $p_1, p_2, \ldots, p_m \models q$, if and only if

$$(\models \mathop{\&}_{i=1}^{m} p_i \Rightarrow q) = (\models \overline{\mathop{\&}_{i=1}^{m} p_i} \vee q) =$$

$(\models \bigvee_{i=1}^{m} \overline{p_i} \vee q)$ $(p_1, p_2, ..., p_m \not\models q$, if and only if

$(\overline{=\!| \mathop{\&}_{i=1}^{m} p_i \Rightarrow q}) = (=\!| \mathop{\&}_{i=1}^{m} p_i \bar{q}))$ [4], where $=\!|$ means „*to be contradictory to*“.



We introduce that all general reasoning $p_1, p_2,...,p_m \models q_1, q_2, ..., q_n$ can be expressed as sequence of Horn's reasoning $p_1, p_2, \ldots, p_m \models q_j$ (j = 1, 2,…, n).

Example 4.: Decide about the correctness of the reasoning

$$\begin{array}{l} \exists x\,(p(x)q(x)) \\ \underline{\forall x\,(p(x) \Rightarrow r(x))} \\ \exists x\,(q(x)r(x)) \end{array}$$

that is, state mapping whether the formula

$$\overline{\exists x}\,(p(x)q(x)) \vee \overline{\forall x}\,(p(x) \Rightarrow r(x)) \vee$$
$$\vee \exists x\,(q(x)r(x)) =$$
$$= \forall x\,\left(\overline{p(x)} \vee \overline{q(x)}\right) \vee \exists x\,\left(p(x)\overline{r(x)}\right) \vee \exists x\,(q(x)r(x))$$

is or is not a tautology:

The above formula is a tautology and the reasoning is correct.

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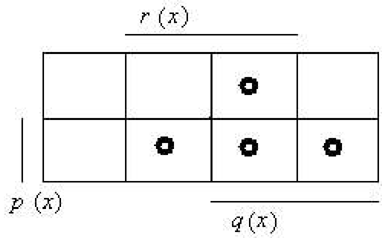
Examle 5.: Verify the correctness of the reasoning

$$\begin{array}{l} \exists x\,(p(x)q(x)) \\ \underline{\exists x\,(p(x)r(x))} \\ \overline{\exists x}\,(q(x)r(x)) \end{array}$$

that is, state mapping whether the formula $\exists x\,(p(x)q(x))\exists x\,(p(x)r(x))\exists x\,(q(x)r(x))$ is a contradiction:



The given formula is not a contradiction and the argument is not correct.

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3. CONCLUSIONS

Authors believe, that the decision of truthfulness values of monadic predicate formulae through Karnaugh maps is very simple.

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BIOGRAPHY

Josef Bokr was born on 1940. In 1965 he graduated with honor at Moscow Power Institute with specialisation in mathematical computing devices and apparatus. He received Ph.D (CSc) degree with a thesis Logic Control in 1990 and was done an associate professor. His scientific research is focused on logic system and automata theory.

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