# ASYMPTOTIC ANALYSIS OF OPTIMAL UNRESTRICTED POLAR QUANTIZATION

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#### **SUMMARY**

The motivation for this work is maintaining high accuracy of phase information that is required for some applications such as interferometry and polarimetry, polar quantization techniques as well as their applications in areas such as computer holography, discrete Fourier transform encoding, and image processing. In this paper the simple and complete asymptotically analysis is given for a nonuniform polar quantizer with respect to the mean-square error (MSE) i.e. granular distortion (Dg). Granular (Support) region (Dependential Reservoirse and Kingdom of a granular considered as the interval of a considered as the interval where  $\frac{1}{2}$  and  $\frac{1}{2}$  and  $\frac{1}{2}$  (Figure ). The motivati Actu Electrotechnica et Information No. 2, Vol. 4, 2004<br> **ASYMPTOTIC ANALYSIS OF OPTIMAL UNRESTRICTED**<br> **POLAR QUANTIZATION**<br> **POLAR QUANTIZATION**<br>
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"Teachly of Electronic Engineering, Interaction Sta **Examplementary and Solution of the Examplementary and Solution of the set of properties of granular (Dependent of Granular (Dependent of Granular (Dependent of Granular Content of Granular (Dependent of the set of granul** combination of granular ( $D_g$ ) and overload ( $D_o$ ) distortions,  $D=D_g+D_o$ . Swaszek and Ku [2] didn't consider the problem of **Figure 14.1**<br> **Example 14.1 ASYMPTOTIC ANALYSIS OF OPTIMAL UNRESTRICTED**<br> **POLAR QUANTIZATION**<br> **POLAR QUANTIZATION**<br> **POLAR QUANTIZATION**<br> **POLAR QUANTIZATION**<br> **POLAR QUANTIZATION**<br> **POLAR QUANTIZATION**<br> **POLAR QUANTIZ in case of the interaction of the CALAMERIAN CONSESS OF OPTIMAL UNRESTRICTED**<br> **POLAR QUANTIZATION**<br> **Case of the corresponding the region of the corresponding to the algorithmic polar and the corresponding that is requi EXECTIVE PERIOD SOLUTIVE AND SOLUTE AND SOLUTE AND SOLUTE CONSEQUENCE CONSEQUENCE (SURVEY IRRESPONDENT CONSEQUENCE THE CONSE** sign to the interval where quantization errors are small, or<br>the interval where quantization errors are small, or<br>product of a quantization errors are small, or<br>product of the total distortion D, which is a<br>Swaszek and Ku

optimum of the polar quantizer and optimal compressor function. The equation for  $D_{\varphi}^{opt}$  is given in a closed form. The

Keywords: phase divisions, number of levels, optimal granular distortion, asymptotical analysis, Unrestricted Polar Quantization

### 1. INTRODUCTION

Polar quantization techniques as well as their applications in areas such as computer holography, discrete Furrier transform encoding, image processing and communications have been studied extensively in the literature. Synthetic Aperture Radars (SARs) images can be represented in the polar format (i.e., magnitude and phase components) [3]. In the case of MSE quantization of a symmetric two-dimensional source, polar quantization gives the best result in the field of the implementation [3]. The motivation behind this work is to maintain high accuracy of phase information that is required for some applications such as interferometry and polarimetry, without loosing massive amounts of magnitude information [3].

One of the most important results in polar quantization are given by Swaszek and Ku who derived the asymptotically Unrestricted Polar Quantization (UPQ) [2]. Swaszek and Ku gave an asymptotic solution for this problem without a mathematical proof of the optimum and using, sometimes, quite hard approximations, which limit to the last physical physical distortion<br>the application polar quantization consists of  $D$ , which is a combination of granular  $(D_g)$  and the application. Polar quantization consists of separate but uniform magnitude and phase quantization, on  $N$  levels, so that rectangular coordinates of the source  $(x,y)$  are transformed into the polar coordinates in the following form:  $r=(x^2+y^2)^{1/2}$ , where r represents magnitude and  $\phi$  is analytical optimal phase:

$$
\phi = \begin{cases} \tan^{-1}\left(\frac{y}{x}\right) \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) \\ \pi + \tan^{-1}\left(\frac{y}{x}\right) \\ 2\pi + \tan^{-1}\left(\frac{y}{x}\right) \end{cases}
$$

for I, II, III and IV quadrant.

The asymptotic optimal quantization problem, even for the simplest case - uniform scalar quantization, is actually nowadays [5]. In [1] the analysis of scalar quantization is done in order to determine the optimal maximal amplitude.

Swaszek and Ku [2] didn't consider the problem of finding the optimal maximal amplitude, so-called, support region.

The support region for scalar quantizers has been found in [1] by minimization of the total distortion  $\phi = \begin{cases} \frac{\sinh 1}{\pi} + \tan^{-1} \left( \frac{y}{x} \right) \\ \frac{1}{\pi} + \tan^{-1} \left( \frac{y}{x} \right) \\ 2\pi + \tan^{-1} \left( \frac{y}{x} \right) \end{cases}$ <br>for I, II, III and IV quadrant.<br>The asymptotic optimal quantization problem,<br>even for the simplest case - uniform scalar<br>qu  $\phi = \begin{cases} \pi + \tan^{-1} \left( \frac{y}{x} \right) \\ \pi + \tan^{-1} \left( \frac{y}{x} \right) \end{cases}$ <br>for I, II, III and IV quadrant.<br>The asymptotic optimal quantization problem,<br>we for the simplest case  $\sim$  uniform scalar<br>quantization, is actually nowadays [5]. overload  $(D_0)$  distortions,  $D=D_g+D_o$ . The goal of this paper is solving the quantization problem in the case of nonuniform polar quantizer and finding the corresponding support region. It is done by analytical optimization of the granular distortion and numerical optimization of the total distortion.

In the paper Peric and Stefanovic [6] analyses are given for optimal asymptotic uniform polar quantization. Analysis of optimal polar quantization Acta Electrotechnica et Informatica No. 2, Vol. 4, 2004<br>
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cuantization. Analysis of optimal polar quantization<br>
for moderate a In this paper the simple and complete asymptotical analyses (for large values N) are given for a nonuniform polar quantizer with respect to the mean-square error (MSE) i.e. granular distortion<br>bivariate density function  $f(x,y) = p(\sqrt{x^2 + y^2})$ .  $(D_g)$ . We consider D as a function of the vector  $\boldsymbol{P}=(P_i)_{1\leq i\leq L}$  whose elements are numbers of phase quantization levels at the each magnitude level. Said by different words, each concentric ring in magnitude is distributed on a  $[0, \infty)$  with density<br>quantization pattern is allowed to have a different function  $f(r) = 2\pi r p(r)$ . Note that magnitude quantization pattern is allowed to have a different number of partitions in the phase quantizer  $(P_i)$ a Electrotechnica et Informatica No. 2, Vol. 4, 2004<br>
In the paper Peric and Stefanovic [6] analyses are<br>
an for optimal asymptotic uniform polar<br>
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moderate and smaller values of when  $r$  is in the *i*-th magnitude ring. Optimal Unrestricted Polar Quantization (OUPQ) must satisfy the constraint  $\sum P_i = N$  in order to use all of 1 L i Informatica No. 2, Vol. 4, 2004<br>
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values of N is given in [7] Stefanovic (6) analyses are **2. CONDITIONS FOR OPTIMALITY AND**<br>mptoinic uniform polar<br>interaction **DESIGNOF UNEESTRECTED POLAR**<br>mptimal polar quantization **QUANTIZER**<br>denote asymptotical and complete asymptotical<br>esc N) a In for optimal asymptotic uniform polar<br>
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 $(r=(r_i)_{1\leq i\leq L+1})$  and number of levels L. We also gave order to mu N regions for the quantization. We prove the variance is:  $2\sigma^2 = 1$ .<br>existence of one minimum and derive the expression We consider nonuniform polar quantizer with L existence of one minimum and derive the expression for evaluating  $P_{opt}(\mathbf{r}, \mathbf{m})$  for fixed values of magnitude le reconstruction levels  $(m=(m_i)_{1\leq i\leq L})$ , decision levels the conditions for optimum of the polar quantizer, optimal compressor function and optimal numbers of levels. We derive  $D_g^{opt}$  in a closed form.

We also gave the example of quantizer  $= r$ ) constructing for a Gaussian source. This case has  $\frac{max}{m}$ . quantizer on an arbitrary source; we can take advantage of the central limit theorem and the known structure of an optimal scalar quantizer for a Gaussian random variable to encode a general process by first filtering it in order to produce an approximately Gaussian density, scalar-quantizing the result, and then inverse-filtering to recover the<br>original [8].<br> $\psi_{i,j} = (2j-1)\pi / P_i$  (see Fig. 1). original [8].

# 2. CONDITIONS FOR OPTIMALITY AND DESIGN OF UNRESTRECTED POLAR **QUANTIZER**

For these analysis we assume that the input is from a continuously valued circularly source with unit variance rectangular coordinate marginals and

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2. CONDITIONS FOR OPTIMALITY AND<br>
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For these analysis we assume that the input is<br>
from a continuously valued circularly source with<br>
unit variance rectangular coordinate margin Transforming to polar coordinates, the phase is uniformly distributed on a  $[0,2\pi)$  and the magnitude is distributed on a  $[0, \infty)$  with density 21<br>
21<br> **DESIGN OF UNRESTRECTED POLAR**<br> **OUANTIZER**<br>
For these analysis we assume that the input is<br>
from a continuously valued circularly source with<br>
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bivariate density and phase are independent random variables. The transformed probability density function for the **21**<br> **OPTIMALITY AND**<br> **EXECTED POLAR**<br> **EXECTED**<br> **EXECTED**<br> **EXECTED**<br> **EXECTED**<br> **EXECTED**<br> **EXECT** 21<br>
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from a continuously valued circularly source with<br>
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Gaussian source is  $f(r, \phi) = \frac{1}{r^2} \cdot re^{\frac{-r^2}{2\sigma^2}}$ 

magnitude levels and  $P_i$  phase reconstruction points at magnitude reconstruction level  $m_i$ ,  $1 \le i \le L$ . In order to minimize the distortion we proceed as follows.

First we partition the magnitude range  $[0,r_{L+1}]$ into magnitude rings by  $L+1$  decision levels (see **Fig. 1)**  $r=(r_1, ..., r_{L+1})$  and  $(0 = r_1 < r_2 < ... < r_L < r_{L+1})$  $= r_{\text{max}}$ ).

the importance because of using Gaussian<br> $m=(m_1,...,m_L)$  obviously satisfy  $(0 \le m_1 \le m_2 \le ...$ The magnitude reconstruction levels (see Fig. 1) Transforming to polar coordinates, the phase is<br>uniformly distributed on a  $[0,2\pi)$  and the<br>magnitude is distributed on a  $[0,\infty)$  with density<br>function  $f(r) = 2\pi r p(r)$ . Note that magnitude<br>and phase are independent rando  $\leq m_L$ ). Next we partition each magnitude ring into  $P_i$  phase subdivisions. Let  $\phi_{ij}$  and  $\phi_{i,j+1}$  be two phase decision levels, and let  $\psi_{i,j}$  be *j*-th phase reconstruction level for the *i*-th magnitude ring,  $1 \leq j$  $P_i$ . Then  $\phi_{i,j} = (j-1)2\pi/P_i$   $j = 1, 2, ..., P_i + 1$ , and probability density function for the<br>
source is  $f(r,\phi) = \frac{1}{2\pi\sigma^2} r e^{\frac{-r^2}{2\sigma^2}} = \frac{f(r)}{2\pi}$ .<br>
oosing generality we assume that<br>  $2\sigma^2 = 1$ .<br>  $2\sigma^2 = 1$ .<br>
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Fig. 1 UPQ and j-th cell on i-th level preview

The distortion D for UPQ  $(r_{L+1} = \infty)$  is [6]:

$$
D = \frac{1}{2} \sum_{i=1}^{L} \sum_{j=1}^{P_i} \int_{\phi_{i,j}}^{\phi_{i,j+1}} [r^2 + m_i^2 - 2rm_i \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi
$$
  
\n(1) 
$$
\frac{\partial^2 D_g}{\partial P_i \partial P_j} = \begin{cases} \frac{\partial^2 D_g}{\partial P_i \partial P_j} & \text{if } i = 1, \text{ and } j = 2, \text{ and } j = 1, \text{ and } j = 2, \text{ and } j = 1, \text{ and } j = 2, \text{ and } j = 1, \text{ and } j = 2, \text{
$$

Total distortion *D*, for OUPQ  $(r_{L+1} = r_{\text{max}})$  is a<br>
combination of granulation and overload distortions  $D=D_g+D_o$ :<br>  $D=D_g+D_o$ : combination of granulation and overload distortions  $D=D_g+D_o$ :

$$
D = \frac{1}{2} \sum_{i=1}^{L} \sum_{j=1}^{P_i} \int_{\phi_{i,j}}^{\phi_{i,j+1}} \int_{r_{i+1}}^{r_{i+1}} [r^2 + m_i^2 - 2rm_i \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi
$$
  
+ 
$$
\frac{1}{2} \sum_{j=1}^{P_i} \int_{\phi_{i,j}}^{\phi_{i,j+1}} \int_{r_{i+1}}^{\phi_{i,j}} [r^2 + m_i^2 - 2rm_i \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi
$$
  
Therefore the formula 
$$
+ \frac{1}{2} \sum_{j=1}^{P_i} \int_{\phi_{i,j}}^{\phi_{i,j+1}} [r^2 + m_i^2 - 2rm_i \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi
$$
  
Therefore the formula 
$$
+ \frac{1}{2} \sum_{j=1}^{P_i} \int_{\phi_{i,j}}^{\phi_{i,j+1}} [r^2 + m_i^2 - 2rm_i \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi
$$
  
(2) constraints 
$$
\sum_{j=1}^{L} P_j = N
$$
. We use the

We integrated (2) by  $\phi$ , and get the equation for granular distortion:

$$
D_g(P_1, \cdots, P_L) = \frac{1}{2} \sum_{i=1}^{L} \int_{r_i}^{r_{i+1}} [r^2 + m_i^2 - 2rm_i \, \text{sinc}(\frac{\pi}{P_i})] f(r) dr \quad (3)
$$

(where in  $sinc(x)=sin(x)/x$ ); (2) we use : and finally:

We integrated (2) by 
$$
\phi
$$
, and get the equation for  
\n
$$
J = D_g + \lambda D_f
$$
, where  $\lambda$  represents  
\n
$$
D_g(P_1, \dots, P_L) = \frac{1}{2} \sum_{i=1}^{L} \int_{\tau_i}^{\tau_i} [r^2 + m_i^2 - 2rm, \sin(\frac{\pi}{P_i})] f(r) dr
$$
\n(3)  $\frac{\partial J}{\partial P_i} = -\frac{2\pi^2}{6(P_i)^3} m_i^2 f(m_i) \Delta_i + \lambda$ ,  
\n(where in sinc(x)=sin(x)/x); (2) we use :  
\n
$$
\frac{\sin(x)}{x} = 1 - \frac{1}{6} x^2 + \varepsilon(x)
$$
\n
$$
D_g \approx \frac{1}{2} \sum_{i=1}^{L} \int_{\tau_i}^{\tau_{i1}} [(r - m_i)^2 + \frac{rm_i}{3} \frac{\pi^2}{P_i^2}] f(r) dr
$$
\n(4) The formula (9) is like to formula in pap  
\nFrom:  $\frac{\partial D_g}{\partial m_i} = 0$   
\nwe can find  $m_i$  as :  
\n
$$
m_i = \left(1 - \frac{1}{6} \frac{\pi^2}{P_i^2}\right) \frac{r_{i+1} + r_i}{2}
$$
\n(5) The approximation given by Swaszek an  
\n(UPQ) [2]:  
\nAs final result, we find approximation for  $m_i$  as:  $r_{L+1} - m_L \approx m_L - r_L = \frac{1}{2Lg'(m_L)}$   
\n(6) is not correct for Unrestricted Polar (1)

From: 
$$
\frac{\partial D_g}{\partial m_i} = 0
$$
 should

we can find  $m_i$  as :

$$
m_{i} = \left(1 - \frac{1}{6} \frac{\pi^{2}}{P_{i}^{2}}\right) \frac{r_{i+1} + r_{i}}{2}
$$
 (5)  $\frac{T}{\text{as}}$ 

As final result, we find approximation for  $m_i$ , as:

$$
m_i = \frac{r_{i+1} + r_i}{2}
$$
 (6) is not correct for Unrestricted Polar Quantization because  $r_{i+1} - m_i \rightarrow \infty$ . That is the elementary

We can obtain from High Resolution Theory [1] that for  $P_i$  satisfy given approximation. high values for  $R$  ( $R = \log_2 N$ ) and critical values

The equation for  $D_g$  is obtained by using High Resolution Theory [6].

$$
D_{g} = \sum_{i=1}^{L} \frac{f(m_{i})\Delta_{i}^{3}}{24} + \sum_{i=1}^{L} \frac{m_{i}^{2} \pi^{2} f(m_{i})\Delta_{i}}{6P_{i}^{2}}
$$
 function, and  
where is  $\Delta_{i} = r_{i+1} - r_{i}$ .

is a convex programming problem. Function  $D<sub>o</sub>(**P**)$ is convex if its Hessian matrix is the positive semidefinite one [4].

[ 2 cos( )] 2 2 i j <sup>r</sup> f r D r m rm drd [ 2 cos( )] <sup>2</sup> i j <sup>r</sup> f r D r m rm drd [ 2 cos( )] <sup>2</sup> <sup>j</sup> <sup>r</sup> f r r m rm drd 2 2 <sup>3</sup> <sup>2</sup> ( ) 6( ) g i i i i <sup>D</sup> m f m P P <sup>i</sup> 2 2 <sup>4</sup> ( ) , ( ) 0, g i i i i i j <sup>D</sup> m f m i <sup>P</sup> P P <sup>2</sup> j i j 2 0 g i j D P P (8) it follows that Dg(P) is a convex function of P. g(P) for fixed

 $2\pi$  The minimization of function  $D_g(P)$  for fixed  $2\pi$  formulated in this way: minimize  $D_g(P)$  under the  $\frac{1}{2}\sum_{i=1}^{L} \sum_{j=1}^{K} \int_{i,j}^{K} [t^2 + m_i^2 - 2m, \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi$ <br>  $\frac{d^2P_z}{\partial P_i \partial P_j} = \begin{cases} \frac{\pi^2}{(R^2)^2} m_i^2 f(m_i) \Delta_{i,1} f = f \\ \frac{\pi^2}{(R^2)^2} m_i^2 f(m_i) \Delta_{i,1} f = f \end{cases}$ <br>
and distortion *D*, for OUPQ  $(r_{z+i} = r_{\text{max}}$  $D = \frac{1}{2} \sum_{i=1}^{k} \sum_{i=1}^{k} \sum_{j=1}^{k} \sum_{j=1}^{k} [1r^2 + m_i^2 - 2rm, cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi$ <br>
For  $D_i$  for  $\text{OUPQ}(r_{i-1} - r_{\text{max}})$  is a<br>
for  $\frac{\partial^2 D_z}{\partial P_i \partial P_j} = \begin{cases} \frac{\pi^2}{(R_i)^2} m_i^2 f(m_i) \Delta_{i,1} = f$ <br>
for  $\frac{\pi^2 D_z}{(R_i)^2} =$ Fotal distortion *D*, for OUPQ  $(r_{i+1}-r_{max})$  is a<br>
combination of granulation and overload distortions<br>  $D=\frac{1}{2}\sum_{i=1}^{k} \sum_{j=1}^{R} \sum_{k=1}^{k} \sum_{k=1}^{k} [r^2 + m_i^2 - 2rm, cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dv d\phi$ <br>  $I = \frac{1}{2}\sum_{i=1}^{k} \sum_{j=1}^{$ on *D*, for OUPQ  $(r_{k+1} = r_{\text{max}})$  is a<br>
f granulation and overload distortions<br>  $\frac{d^2D_g}{dP_l dP_l} \ge 0$ <br>
it follows that  $D_g(P)$  is a convex function of<br>  $[r^2 + m_k^2 - 2rm, \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi$ <br>
interior of magnitude le (8)<br>  $\sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{j=1}^{n} [r^2 + m_i^2 - 2m, \cos(\phi - \psi_{r,j})] \frac{f(r)}{2\pi} dr d\phi$  it follows that  $D_g(P)$  is a convex function of P.<br>
The minimization of function  $D_g(P)$  is  $D_g(P)$  is  $D_g(P)$  is  $D_g(P)$  for fixed<br>  $\sum_{j=1}^{n} \sum_{$  $D = D_{\phi} + D_{\phi}$ . (8)<br>  $D = \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{n} \sum_{k=1}^{i} [r^2 + m_i^2 - 2m_i \cos(\phi - \psi_{i,j})] \frac{f(r)}{2\pi} dr d\phi$  it follows that  $D_{\phi}(P)$  is a convex function of  $P$ .<br>
The minimization of function  $D_{\phi}(P)$  for fixed<br>  $-\frac{1}{2} \sum$ (8)<br>  $\phi P_{\nu}(P_{f})$ <br>
it follows that  $D_{g}(P)$  is a convex function of *P*.<br>  $\phi P_{\nu}(P_{f})$ <br>  $\frac{d}{dx}dr d\phi$ <br>
The minimization of function  $D_{g}(P)$  for fixed<br>
number of magnitude levels *L* constrained by the<br>
total number of We integrated (2) by  $\phi$ , and get the equation for<br>  $J = D_g + \lambda \sum P_g$ . We use the<br>
granular distortion:<br>  $D_g(P_i, \dots, P_i) = \frac{1}{2} \sum_{i=1}^{L} \frac{1}{2} [r^2 + m_i^2 - 2m_i \sin(\frac{\pi}{P_i})] I(r) dr$ <br>  $D_g(P_i, \dots, P_i) = \frac{1}{2} \sum_{i=1}^{L} \frac{1}{2} [r^2 + m_i^2 - 2m$ number of magnitude levels L constrained by the total number of reconstruction points  $N$  is 1 L i  $\frac{2\pi^2}{(P_i)^3} m_i^2 f(m_i) \Delta_i$ <br>  $\frac{\pi^2}{(P_i)^3} m_i^2 f(m_i) \Delta_i$ ,  $i = j$ <br>  $i ≠ j$ <br>  $i ≠ j$ <br>  $i ≠ j$ <br>  $i ≠ j$ <br>  $P_i ∂ P_j$ <br>  $P_i ∂ P_j$ <br>  $\ge 0$ <br>
(8)<br>  $D_g(P)$  is a convex function of *P*.<br>
Attion of function  $D_g(P)$  for fixed<br>
agnitude levels *L*  $J=D_g+\lambda \sum P_i$ , where  $\lambda$  represents Lagrange multiplier. From  $\frac{\partial J}{\partial P_i} = 0$  we obtain : 2  $G(P_i)^{3m_i}$   $J(m_i) \Delta_i$ ,  $i = j$ <br>  $\begin{cases} \frac{\pi^2}{(P_i)^4} m_i^2 f(m_i) \Delta_i$ ,  $i = j \end{cases}$ <br>  $\Rightarrow \frac{\partial^2 D_g}{\partial P_i \partial P_j} \ge 0$  (8)<br>
s that  $D_g(P)$  is a convex function of P.<br>
inimization of function  $D_g(P)$  for fixed<br>
of manitude levels L constrain  $\frac{\partial^2 D_g}{\partial P_i} = \frac{\frac{\partial^2 D_g}{\partial P_i \partial P_j} = \begin{cases} \frac{\partial^2 D_g}{\partial P_i \partial P_j} & \text{if } j \ (m_i) \Delta_i, i = j \end{cases}$ <br>  $\Rightarrow \frac{\partial^2 D_g}{\partial P_i \partial P_j} = 0$  (8)<br>
follows that  $D_g(P)$  is a convex function of P.<br>
ne minimization of function  $D_g(P)$  for fixed<br>
member of  $\frac{\partial^2 D_g}{\partial P_i \partial P_j} = \begin{cases} \frac{\pi^2}{(R_i)^4} m_i^2 f(m_i) \Delta_i, i = j \\ \frac{\partial^2 D_g}{\partial P_i \partial P_j} = 0 \end{cases}$ <br>
Sollows that  $D_g(P)$  is a convex function of *P*.<br>
(8)<br>
follows that  $D_g(P)$  is a convex function of *P*.<br>
ne minimization of function  $D_g(P)$ 0,  $i \neq j$ <br>  $\frac{\partial^2 D_g}{\partial P_i \partial P_j} \geq 0$  (8)<br>
at  $D_g(\mathbf{P})$  is a convex function of P.<br>
ization of function  $D_g(\mathbf{P})$  for fixed<br>
magnitude levels L constrained by the<br>
in this way: minimize  $D_g(\mathbf{P})$  under the<br>
in this way (8)<br>
nvex function of *P*.<br>
unction  $D_g(P)$  for fixed<br>
els *L* constrained by the<br>
truction points *N* is<br>
inimize  $D_g(P)$  under the<br>
We use the equation:<br>
represents Lagrange<br>
we obtain :<br>  $+\lambda$ ,<br>  $\lambda$ ,<br>  $1 \le i \le L$ . (9)<br>
orm  $\frac{1}{\sqrt{2}} \ge 0$  (8)<br>
(8)<br>
(8)<br>
(8)<br>
(8)<br>
(8)<br>
(6)<br>
function  $D_g(P)$  for fixed<br>
ude levels L constrained by the<br>
reconstruction points N is<br>
way: minimize  $D_g(P)$  under the<br>  $=N$ . We use the equation:<br>
ere  $\lambda$  represents La  $i \neq j$ <br>  $\frac{\partial^2 D_g}{\partial P_i \partial P_j} \geq 0$  (8)<br>  $D_g(P)$  is a convex function of *P*.<br>
ation of function  $D_g(P)$  for fixed<br>
agnitude levels *L* constrained by the<br>
t of reconstruction points *N* is<br>
this way: minimize  $D_g(P)$  under t  $\Rightarrow \frac{\partial^2 D_g}{\partial P_i \partial P_j} \ge 0$  (8)<br>
t follows that  $D_g(P)$  is a convex function of *P*.<br>
the minimization of function  $D_g(P)$  for fixed<br>
umber of magnitude levels *L* constrained by the<br>
both umber of reconstruction points *N*  $\frac{L_g}{m_i^2 P_j} \ge 0$  (8)<br>  $D_g(P)$  is a convex function of *P*.<br>
(8)<br>
tion of function  $D_g(P)$  for fixed<br>
of reconstruction points *N* is<br>
this way: minimize  $D_g(P)$  under the<br>
this way: minimize  $D_g(P)$  under the<br>
where  $\lambda$  r under or magnitude levels L construction points N<br>
interact and a number of reconstruction points N is<br>
by informulated in this way: minimize  $D_g(P)$  under the<br>
nonstraints  $\sum_{i=1}^{L} P_i = N$ . We use the equation:<br>  $= D_g + \lambda \sum$ unimpler of magnitude levels L constrained by the<br>
dotal number of reconstruction points N is<br>
ormulated in this way: minimize  $D_g(P)$  under the<br>
constraints  $\sum_{i=1}^{L} P_i = N$ . We use the equation:<br>  $T=D_g + \lambda \sum P_i$ , where  $\lambda$ 

(3) 
$$
\frac{\partial J}{\partial P_i} = -\frac{2\pi^2}{6(P_i)^3} m_i^2 f(m_i) \Delta_i + \lambda ,
$$
  
and finally:

multiplier. From 
$$
\frac{\partial J}{\partial P_i} = 0
$$
 we obtain:  
\n
$$
\frac{\partial J}{\partial P_i} = -\frac{2\pi^2}{6(P_i)^3} m_i^2 f(m_i) \Delta_i + \lambda,
$$
\nand finally:  
\n
$$
P_{iopt} = N \frac{\sqrt[3]{m_i^2 f(m_i) \Delta_i}}{\sum_{j=1}^{L} \sqrt[3]{m_j^2 f(m_j) \Delta_j}};
$$
\n $1 \le i \le L$ . (9)  
\nThe formula (9) is like to formula in paper [7] (i.e. it should obtained utilizing approximation  
\n
$$
\int_{\tau_i}^{\tau_{i+1}} f(f(r)) dr \approx m_i f(m_i) \Delta_i ).
$$
\nThe approximation given by Swaszek and Ku for the asymptotically. Unrestricted Polar Quantization (UPQ) [2]:  
\n $r_{L+1} - m_L \approx m_L - r_L = \frac{1}{2Lg'(m_L)}$  (10)  
\nis not correct for Unrestricted Polar Quantization  
\nbecause  $r_{L+1} - m_L \rightarrow \infty$ . That is the elementary  
\nreason for introducing support region ( $r_{\text{max}}$ ), where

The formula (9) is like to formula in paper [7] (i.e. it should obtained utilizing approximation

$$
\int_{r_i}^{r_{i+1}} r f(r) dr \approx m_i f(m_i) \Delta_i).
$$

 $\frac{a}{b}$  (5) The approximation given by Swaszek and Ku for the asymptotically Unrestricted Polar Quantization The approximation given by Swaszek and Ku for the (UPQ) [2]:

$$
r_{L+1} - m_L \approx m_L - r_L = \frac{1}{2Lg'(m_L)}
$$
 (10)

Where  $\lim_{x \to 0} \frac{\sin(x)}{x} = 1 - \frac{1}{6}x^3 + \varepsilon(x)$ <br>  $\lim_{x \to \infty} \frac{1}{2} \sum_{r=1}^{r} \frac{r_1}{r_2} (r - m_r)^2 + \frac{rm_r}{3} \frac{\pi^2}{P_r^2} \int_0^r r_1 r_2 dr$ <br>
From:  $\frac{\partial D_\varepsilon}{\partial m_r} = 0$ <br>
From:  $\frac{\partial D_\varepsilon}{\partial m_r} = 0$ <br>
we can find *m*, as:<br>  $m_r = \left(1 - \frac$  $\frac{\partial D_x}{\partial H_1} = 0$ <br>
The formula (9) is like to formula in paper [7] (i.e. it<br>  $\frac{\partial D_x}{\partial H_2} = 0$ <br>
find *m*, as:<br>  $\int_0^{x_1} f'(r) dr \approx m_i f(m_i) \Delta_i$ .<br>
Find *m*, as:<br>  $\int_0^{x_2} f'(r) dr \approx m_i f(m_i) \Delta_i$ .<br>
In  $\frac{1}{6} \frac{\pi^2}{P_i^2} \Big|_0^{x_{i=$ = 0<br>
= 0<br>
m, as:<br>  $\frac{\pi^2}{P_r^2} \sum_{l=1}^{n} \frac{r_{l-1} + r_l}{2}$ <br>
(5) and obtained utilizing approximation<br>  $m_i$  as:<br>  $\int_{r_i}^{2\pi} f(r) dr \approx m_i f(m_i) \Delta_i$ .<br>
The approximation given by Swaszek and Ku for<br>  $\frac{\pi^2}{P_r^2} \sum_{l=1}^{n} \frac{r_{l-$ From:  $\frac{\partial L_e}{\partial m_i} = 0$ <br>
From:  $\frac{\partial L_e}{\partial m_i} = 0$ <br>
Should obtained utilizing approximation<br>
we can find  $m_i$  as:<br>  $m_i = \left(1 - \frac{1}{6} \frac{\pi^3}{P_i^2}\right) \frac{r_{i+1} + r_i}{2}$ <br>
(5) The spressimal result, we find approximation for  $m_i$  as From:  $\frac{1}{\omega m_i} = 0$ <br>
we can find m, as:<br>  $m_i = \left(1 - \frac{1}{6} \frac{\pi^2}{P_i^2}\right) \frac{r_{i-1} + r_i}{2}$  (5) The approximation given by Swaszek and<br>
As final result, we find approximation for  $m_i$  as:<br>  $\frac{r_{i-1} - m_i \approx m_i - r_i}{2} = \frac{1}{2Lg'(m$ Co we obtain :<br>
(a) Δ<sub>i</sub> + λ,<br>
(b) Δ<sub>j</sub> : 1 ≤ i ≤ L. (9)<br>
(b) formula in paper [7] (i.e. it<br>
ig approximation<br>
(c).<br>
(c).<br>
(c).<br>
(c).<br>
(c).<br>
(d) Dar Quantization<br>  $\frac{1}{Lg^{c}(m_L)}$  (10)<br>
testricted Polar Quantization<br>  $\in$  $\frac{\partial P}{\partial P_i} = -\frac{\delta(P_i)^+}{6(P_i)^+} m_i^+ J(m_i) \Delta_i + \lambda$ ,<br>
and finally:<br>  $P_{\text{ryst}} = N \frac{\sqrt{m_i^2 f(m_i) \Delta_i}}{\sum_{j=1}^{i} \sqrt{m_j^2 f(m_j) \Delta_j}};$   $1 \le i \le L$ . (9)<br>
The formula (9) is like to formula in paper [7] (i.e. it<br>
should obtained utilizing appr and finally:<br>  $P_{\text{top}} = N \frac{\sqrt{m_i^2 f(m_i) \Delta_i}}{\sum_{j=1}^{L} \sqrt[3]{m_j^2 f(m_j) \Delta_j}};$  1 ≤ *i* ≤ *L*. (9)<br>
The formula (9) is like to formula in paper [7] (i.e. it<br>
should obtained utilizing approximation<br>  $\int_{\tau_1}^{\tau_2} f(f(r)) dr \approx m_i f(m_i) \Delta_i$  $r_{\text{max}}$  is restricted for the scalar quantization analysis, which is based on using compressor function g. reason for introducing support region ( $r_{\text{max}}$ ), where  $n_j^r J(m_j) \Delta_j$ <br>
is like to formula in paper [7] (i.e. it<br>
utilizing approximation<br>  $f(m_i) \Delta_i$ ).<br>
ion given by Swaszek and Ku for the<br>
Unrestricted Polar Quantization<br>  $r_L = \frac{1}{2Lg \cdot (m_L)}$  (10)<br>
for Unrestricted Polar Quantiz  $(m_j)\Delta_j$ <br>
ike to formula in paper [7] (i.e. it<br>
izing approximation<br>  $\sum_i \Delta_i$ ).<br>
given by Swaszek and Ku for the<br>
interstricted Polar Quantization<br>  $=\frac{1}{2Lg'(m_L)}$  (10)<br>
Unrestricted Polar Quantization<br>  $\rightarrow \infty$ . That is the dr  $\approx m_i f(m_i) \Delta_i$ .<br>
or  $\approx m_i f(m_i) \Delta_i$ .<br>
Troximation given by Swaszek and Ku for the<br>
trivially Unrestricted Polar Quantization<br>
21:<br>  $\approx m_L - r_L = \frac{1}{2Lg \cdot (m_L)}$  (10)<br>
correct for Unrestricted Polar Quantization<br>  $r_{L+1} - m_L \rightarrow \in$  $f(m_i) \Delta_i$ .<br>
tion given by Swaszek and Ku for the<br>
Unrestricted Polar Quantization<br>  $-r_L = \frac{1}{2Lg \cdot (m_L)}$  (10)<br>
for Unrestricted Polar Quantization<br>  $m_L \rightarrow \infty$ . That is the elementary<br>
ducing support region  $(r_{\text{max}})$ , where<br> <sup>2</sup><br>  $\int_{1}^{4} rf(r) dr \approx m_{i} f(m_{i}) \Delta_{i}$ .<br>
he approximation given by Swaszek and Ku for the<br>
2PO) [2]:<br>
Unrestricted Polar Quantization<br>
2PO) [2]:<br>  $\int_{1}^{4} r^{4}(-n_{i} - m_{i} \approx m_{i} - r_{i} = \frac{1}{2Lg'(m_{i})}$  (10)<br>
s not correct for Unr  $r_j dr \approx m_{i,j} (m_i) \Delta_i$ ).<br>
pproximation given by Swaszek and Ku for the<br>
obtotically Unrestricted Polar Quantization<br>  $|2]$ :<br>  $m_L \approx m_L - r_L = \frac{1}{2Lg \cdot (m_L)}$  (10)<br>
ot correct for Unrestricted Polar Quantization<br>
se  $r_{L+1} - m_L \rightarrow \infty$ 

 $\overline{P_i^2}$  (*i*)  $(\Delta_i \approx dr)$ , and we get  $P_i$  as: We replaced  $\Delta_i = \frac{r_{\text{max}}}{r_{\text{max}}}$ , where g is compressor function, and approximate the sums by integrals

$$
P_i \approx \frac{Nr_{\text{max}} \sqrt[3]{m_i^2 f(m_i)/g'(m_i)}}{L \int_0^{r_{\text{max}}} \sqrt[3]{r^2 f(r)(g'(r))^2} dr}.
$$
 (11)

As final result, we find the equation for granular distortion:

\n
$$
D_g = \frac{r_{\text{max}}^2}{24L^2} \int_0^{r_{\text{max}}} \frac{f(r)}{(g'(r))^2} dr + \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} \left( \int_0^3 \sqrt[3]{r^2 f(r)(g'(r))^2} dr \right) = \frac{r_{\text{max}}^2}{24L^2} \int_0^{\frac{r_{\text{max}}}{2}} \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2}
$$
\n

\n\n
$$
= \frac{r_{\text{max}}^2}{24L^2} \int_0^{\frac{r_{\text{max}}}{2}} \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} \left( \int_0^3 \sqrt[3]{r^2 f(r)(g'(r))^2} dr \right)^3 = \frac{r_{\text{max}}^2}{24L^2} \int_0^1 \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} \int_0^3 \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2}
$$
\n

\n\n
$$
= \frac{r_{\text{max}}^2}{24L^2} \int_0^1 \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} \int_0^3 \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} \int_0^3 \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2}
$$
\n

\n\n
$$
= \frac{r_{\text{max}}^2}{6L^2} \int_0^1 \frac{\pi^2 L^2}{4L^4} \int_0^1 \frac{\pi^2 L^2}{3N^2 r_{\text{max}}^2} \int_0^3 \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} \int_0^3 \frac{\pi
$$

 $\frac{{}^2D_g}{{}^{2}T^2} = \frac{r_{\text{max}}^2}{4T^4}I_0 + \frac{\pi^2}{3M^2r^2}I^3$ rotechnica et Informatica No. 2, Vol. 4, 2004<br>  $\sum_{i=1}^{n} \frac{f(r)}{g(r)} dr$ <br>  $\frac{D_g}{2\pi r^2} = \frac{r_{\text{max}}^2}{r^2} I_0 + \frac{\pi^2}{2\pi^2 r^2} I^3$ . The optimal number With goal to hnica et Informatica No. 2, Vol. 4, 2004<br>  $\frac{f(r)}{(g'(r))^2} dr$   $\qquad$ <br>  $\frac{f(r)}{(g'(r))^2} dr$   $\qquad$ <br>  $\frac{f(r)}{(g'(r))^2} dr$   $\qquad$ <br>  $\frac{f(r)}{(g'(r))^2} dr$   $\qquad$ <br>  $\qquad$  This follows from the fact that r and m are equal<br>  $\frac{f(r)}{(g'(r))^2} dr$   $\qquad$ <br>  $\$ a Electrotechnica et Informatica No. 2, Vol. 4, 2004<br>  $=\frac{r_{\text{max}}^2}{24L^2} \int_0^{r_m} \frac{f(r)}{g(rr)^2} dr$ <br>  $+\frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} \int_0^{r_m} \sqrt{r^2 f(r)(g'(r))^2} dr$ <br>  $+\frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} \int_0^{r_m} \sqrt{r^2 f(r)(g'(r))^2} dr$ <br>  $=\frac{r_{\text{max}}^2}{24L^$ of levels problem can be solved analytically only for the asymptotical analysis as it is suggested: from the condition  $\frac{\partial D_s}{\partial s} = 0$  we came to the optimal solution  $\frac{\epsilon}{L} = 0$  we came to the optimal solution For estimation Example a thromatica No. 2, Vol. 4, 2004<br>  $\int_{0}^{\infty} \frac{f(r)}{(g'(r))^2} dr$ <br>  $\int_{0}^{\infty} \frac{f(r)}{(g'(r))^2} dr$ <br>  $\int_{0}^{\infty} \frac{f(r)}{(g'(r))^2} dr$ <br>  $\int_{0}^{\infty} \frac{f(r)}{6N^2 r_{max}^2} I^3$ <br>  $\int_{0}^{\infty} \frac{d^2r^2}{r^2} dr$ <br>  $\int_{0}^{\infty} \frac{d^2r^2}{r^2} dr$ <br>  $\$  $D_{\rm g} = \frac{r_{\rm max}^2}{24L^2} \int_0^{\infty} \frac{f(r)}{(g(r))^2} dr +$ <br>  $+\frac{\pi^2 L^2}{6N^2 r_{\rm max}^2} \int_0^{\infty} \frac{1}{3} \sqrt{r^2 f(r)(g(r))^2} dr^3 =$ <br>  $-\frac{r_{\rm max}^2}{24L^2} I_0 + \frac{\pi^2 L^2}{6N^2 r_{\rm max}^2}$ <br>  $-\frac{r_{\rm max}^2}{24L^2} I_0 + \frac{\pi^2 L^2}{6N^2 r_{\rm max}^2} I^3$ <br>
(12) th  $rac{x^2 L^2}{\sqrt{t^2}}$   $rac{v_{\text{max}}}{\sqrt{t^2}}$   $\left(\int_0^{\pi/2} \sqrt{t^2/(t^2/(t^2))^2} dt\right)^3 =$ <br>  $\left(\int_0^{\pi/2} \sqrt{t^2/(t^2$ (a)  $\left(-\int_{0}^{\infty} \sqrt[3]{r^2 f(r)(g'(r))^2} dr \right)^3 =$ <br>  $\left(-\int_{0}^{\infty} \sqrt[$ is dependent of  $m$ ,  $N$  and introduced approx<br>  $= \frac{r_{\text{max}}^2}{24L^2} I_0 + \frac{\pi^2 L^2}{6N^2 r_{\text{max}}^2} I^3$  (12) then  $\sum_{i=1}^{k} P_i = N$  will not be satisfied. In ad<br>
some values of  $N$  from the former range,<br>
The function  $D_4(L)$ 24 $L^{2-6}$  6  $N^2 r_{\text{max}}^2$  some values of *N* from the former range, we cannot<br>
The function  $D_g(U)$  is convex of *L*, because<br>  $\sum_{i=1}^{n} P_i = N$ .<br>  $\frac{r_{\text{min}}^2}{2L^2} = \frac{r_{\text{max}}^2}{4L^2} I_0 + \frac{\pi^2}{3} N^2 r_{\text{max}}^2 I^2$ . The o 24 $L^{2-10}$  6 $N^2 r_{\text{cm}}^2$ <br>  $\frac{d^2 L^2}{dt^2} = \frac{r_{\text{cm}}^2}{r_{\text{cm}}^2} I^2$ . The optimal number<br>  $\frac{d^2 L^2}{dt^2} = \frac{r_{\text{cm}}^2}{r_{\text{cm}}^2} I^2$ . The optimal number<br>  $\frac{d^2 L^2}{dt^2} = \frac{r_{\text{cm}}^2}{4t^2} I^2 + \frac{r_{\text{cm}}^2}{3N^2 r_{\text{cm}}$  $\frac{d}{dx}$   $D_g(L)$  is convex of L, because reach  $\sum_{r=1}^{L} P_i = N$ .<br>  $\frac{\pi^2}{3N^2 r_{\text{max}}^2} I^3$ . The optimal number With goal to calculate rough (approximately) the deviation of calculated number of points than analysis as i =  $\frac{r_{\text{max}}^2}{\sqrt{t^2 + r^2}}$   $\int$   $\frac{r}{3N^2 r_{\text{max}}^2}$   $I^2$ . The optimal number<br>
4 styroblem can be solved analytically only for<br>
the goal to calculate rough (approximate<br>  $r_{\text{approx}}$  for belowing to the primal solution o

for  $L_{opt}$ :

$$
L_{opt} = r_{\text{max}} \sqrt[4]{\frac{I_0 N^2}{4\pi^2 I^3}}
$$
 (13) 
$$
\sum_{i=1}^{L} P_i = \sum_{i=1}^{L} \sqrt{\pi}
$$

The optimal granular distortion is:

$$
D_g^{\text{opt}} = \frac{\pi}{6N} I \sqrt{I_0 I} \tag{14} \approx r
$$

inequality:

$$
g(r) = \left(r_{\text{max}} \int_{0}^{r} \sqrt[4]{\frac{f(r)}{r}} dr\right) / \left(\int_{0}^{r_{\text{max}}} \sqrt[4]{\frac{f(r)}{r}} dr\right) \tag{15}
$$

and 
$$
D_g^{\text{opt}} = \frac{\pi}{6N} \left( \int_0^{r_{\text{max}}} \sqrt{rf(r)} dr \right)^2.
$$
 (16) We can  

$$
N = M =
$$

#### Example:

We compared results for Gaussian source. Numbers of magnitude levels and reconstruction points, reconstruction points and decision levels are calculated by using (for Gaussian source [2]):

$$
L = \sqrt{N/2}
$$
\n
$$
I = \sqrt{\pi} N^{1/2} m_i \exp(-\frac{m_i^2}{8})
$$
\n
$$
r_i = g^{-1}[(i-1)/L], 1 \le i \le L; \quad r_{L+1} = \infty
$$
\nCorrect analysis, i.e., number of points that will give for L=11, i. N=10.

$$
m_i = g^{-1}[(2i-1)/2L], 1 \le i \le L
$$

 $g(r)$  is a compressor function given by :  $\qquad \qquad \text{are}$ 

$$
g(r) = \left(\int_0^r \sqrt{\frac{f(s)}{s}} ds\right) / \left(\int_0^{\infty} \sqrt{\frac{f(s)}{s}} ds\right)
$$

Method presented in the paper  $[2]$  cann't be applied for some values of  $N$  and numbers of level  $L$ . For number of level  $L$ , the total number of points is in the range,

23<br>  $(\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor), N_1 = 2(*rand*(L) - 0.5)<sup>2</sup>, N_2 = 2(*rand*(L) + 0.5)<sup>2</sup>.$ <br>
This follows from the fact that **r** and **m** are equal for<br>
any N in the range( $\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor$ ), and since  $P_{opt}$ <br>
is dependen  $(\lceil N_1 \rceil \le N \le N_2 \rceil), N_1 = 2$ *round*(*L*)-0.5<sup>2</sup>,  $N_2 = 2$ *round*(*L*)+0.5<sup>2</sup>.<br>This follows from the fact that *r* and *m* are equal for any *N* in the range( $\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor$ ), and since  $P_{opt}$  $(\lceil N_1 \rceil \le N \le N_2 \rceil), N_1 = 2(\text{rand}(L) - 0.5)^2, N_2 = 2(\text{rand}(L) + 0.5)^2$ .<br>This follows from the fact that r and m are equal for 23<br>  $N_2 = 2$ *round*(*L*)+0.5<sup>2</sup>.<br>
and *m* are equal for<br>  $2 \int$ ), and since  $P_{opt}$ <br>
ded approximations,<br>
ed. In addition, for 23<br>  $\text{Normal}(L) - 0.5)^2$ ,  $N_z = 2\text{rand}(L) + 0.5)^2$ .<br>
in e fact that r and m are equal for<br>  $N_1 \ge N \le \lfloor N_2 \rfloor$ ), and since  $P_{opt}$ <br>
and introduced approximations,<br>
not be satisfied. In addition, for<br>
om the former range, we canno <sup>23</sup><br>
( $\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor$ ),  $N_1 = 2$ (*rand*(*L*)-0.5<sup>2</sup>,  $N_2 = 2$ /*rand*(*L*)+0.5<sup>2</sup>.<br>
This follows from the fact that *r* and *m* are equal for<br>
any *N* in the range( $\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor$ ), and since  $P_{opt}$ <br>
i

1 L i <sup>23</sup><br>  $iN \leq [N_2]$ ),  $N_1 = 2$ (round(L)-0.5<sup>2</sup>,  $N_2 = 2$ (round(L)+0.5<sup>2</sup>.<br>
billows from the fact that **r** and **m** are equal for<br>
in the range( $\lceil N_1 \rceil \leq N \leq \lfloor N_2 \rfloor$ ), and since  $P_{opt}$ <br>
endent of **m**, N and introduced some values of N from the former range, we cannot reach  $\sum P_i = N$ . 1 L i 23<br>  $([\![N]\!] \leq N \leq [\![N_2]\!])$ ,  $N_1 = 2$ *(rand(L)*-0.5)<sup>2</sup>,  $N_2 = 2$ *(rand(L)*+0.5)<sup>2</sup>.<br>
This follows from the fact that *r* and *m* are equal for<br>
any *N* in the range( $[\![N_1]\!] \leq N \leq [\![N_2]\!]$ ), and since  $P_{opt}$ <br>
is dependent  $|N_i| \le N \le |N_2|$ ,  $N_i = 2\pi \text{ and } (D-0.5)^2$ ,  $N_i = 2\pi \text{ and } (D+0.5)^2$ .<br>
his follows from the fact that  $r$  and  $m$  are equal for<br>
ny  $N$  in the range( $\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor$ ), and since  $P_{opt}$ <br>
dependent of  $m$ ,  $N$  and intr  $\begin{aligned}\n\frac{1}{N}\left|\leq N\leq N_{\pm}\right|N_{\pm}\left|N_{\pm}\right|, N_{\pm} = 2\sqrt{\pi}\text{arct}(L) - 0.5)^2, N_{\pm} = 2\sqrt{\pi}\text{arct}(L) + 0.5)^2.\n\end{aligned}$ s follows from the fact that **r** and **m** are equal for  $N$  in the range( $\left[N_1\right] \leq N \leq \left[N_2\right]$ ), and since  $P_{$ bows from the rate that r and *m* are equal for<br>the range( $\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor$ ), and since  $P_{opt}$ <br>dent of *m*, *N* and introduced approximations,<br> $P_{opt}$ <br> $P_i = N$  will not be satisfied. In addition, for<br>thus of *N* from Its books from the later that *a* and *m* and *m* are equal for<br>
y N in the range( $\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor$ ), and since  $P_{opt}$ <br>
dependent of *m*, N and introduced approximations,<br>  $\sum_{i=1}^{L} P_i = N$  will not be satisfied. I  $\int_{1}^{R} P_i = N$  will not be satisfied. In addition, for<br>
lulues of N from the former range, we cannot<br>
lulues of N from the former range, we cannot<br>
and to calculate rough (approximately) the<br>
n of calculated number of poi en  $\sum_{i=1}^{L} P_i = N$  will not be satisfied. In addition, for<br>me values of *N* from the former range, we cannot<br>ach  $\sum_{i=1}^{L} P_i = N$ .<br>it and to calculate rough (approximately) the<br>viation of calculated number of points than

deviation of calculated number of points than proposed number of points N by the method from paper [2], we will make next approximate analisys.

For estimation of  $\sum P_i$  we gave 1 L  $\sum_{i=1}^{n} P_i$  we gave following approximation: we found the total number of points [2] as:

The function D<sub>g</sub>(L) is convex of L, because  
\n
$$
\frac{\partial^2 D_g}{\partial L^2} = \frac{r_{\text{max}}^2}{4L^4} I_0 + \frac{\pi^2}{3N^2 r_{\text{max}}^2} I^3.
$$
\nThe optimal number  
\nWiel goal to calculate rough (approximately) the  
\nderible problem can be solved analytically only for  
\nthe asymptotical analysis as it is suggested: from the  
\nthe asymptotential analysis as it is suggested: from the  
\n*long*:  
\n
$$
\frac{\partial D_g}{\partial L} = 0
$$
\nwe came to the optimal solution  
\nFor  $L_{opt}$ :  
\n
$$
L_{opt} = r_{\text{max}} \sqrt{\frac{I_0 N^2}{4 \pi^2 I^2}}
$$
\n(13)\n
$$
\sum_{i=1}^{k} P_i = \sum_{i=1}^{k} \sqrt{\pi N m_i} \exp(-\frac{m_i^2}{8}) \Delta_i \approx
$$
\nThe optimal solution  
\nWe can obtain *g*(*r*) like in [2] by using Hölder's  
\nnegative  
\ninequality:  
\n
$$
g(r) = (r_{\text{max}} \int_0^{\pi} \sqrt{\frac{f(v)}{r}} dr) / (\int_0^{\pi} \sqrt{\frac{f(r)}{r}} dr) \qquad (15)
$$
\n
$$
= round(L) \sqrt{\frac{N}{2}} \int_0^{\pi} r \exp(-\frac{r^2}{4}) dr =
$$
\n
$$
g(r) = (r_{\text{max}} \int_0^{\pi} \sqrt{\frac{f(v)}{r}} dr) / (\int_0^{\pi} \sqrt{\frac{f(r)}{r}} dr) \qquad (15)
$$
\nand 
$$
D_{\tilde{g}}^{\text{pre}} = \frac{\pi}{6N} (\int_0^{\pi} (\int_0^{\pi} \sqrt{r(r)} dr)^2).
$$
\n(d) We considered the most critical values for  
\nmeasurable:  
\n
$$
\delta_i = |M_i - M|
$$
 (see Table 1.)  
\nWe compared results for Gaussian source. Numbers  
\nreconstrained points and decision levels are  
\n
$$
L = \sqrt{N/2}
$$
\n
$$
L = \sqrt{N/2}
$$
\n
$$
P_i = \sqrt{\pi} N^{1/2} m_i \exp(-\frac{m_i^2}{8})
$$
\nTable 2.

$$
(15) \qquad \qquad \overline{\qquad \qquad \cdots \qquad \qquad \cdots}
$$

We considered the most critical values for  $N=M_1=\begin{bmatrix} N_1 \end{bmatrix}$  and  $N=M_2=\begin{bmatrix} N_2 \end{bmatrix}$  where  $\delta_i = |M_i - M|$ . (see Table 1.)



# Table 1.

Correct analysis i.e the deviation of calculated number of points than proposed number of points we will give for  $L=11$  i  $N=221$ . (see Table 2.)

By Swaszek and Ku [2] for each L=const, m and r are equal. For  $N = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = 221 \Rightarrow L=11$ 

$$
\sum_{i=1}^{L} P_i = 232,84, \text{ and } \delta_1 = 11,84 \text{ (approximately)}
$$

 $\delta_1 = 10,26$  from Table 1).

For  $P_i=round(P_i)$  we can't satisfy constraint L i rounding, but 9 of them are different from values in [2].



Table 2.

$$
L_{opt} = r_{\text{max}} \sqrt[4]{\frac{I_0 N^2}{4\pi^2 I^3}} \; ;
$$

 $g(r)$  is a compressor function given by :

$$
g(r) = (r_{\text{max}} \int_0^r \sqrt[4]{\frac{f(r)}{r}} dr) / \left(\int_0^{r_{\text{max}}} \sqrt[4]{\frac{f(r)}{r}} dr\right)
$$

$$
P_{iopt} = N \frac{\sqrt[3]{m_i^2 f(m_i) \Delta_i}}{\sum_{j=1}^L \sqrt[3]{m_j^2 f(m_j) \Delta_j}}; \qquad 1 \le i \le L
$$

Step 3)

The exact optimal value for  $r_{\text{max}}$  is obtained repeating our optimization method for different  $r_{\text{max}}$ and choosing the values for which  $D = D_g + D_o$  is minimal.

## 3. CONCLUSION

The solution given by Swaszek and Ku[2] is the best one found by now but for large N. Swaszek and Ku gave an asymptotic solution for unrestricted nonuniform polar quantization without a mathematical proof of the optimum and using, sometimes, quite hard approximations, which limit the application. We gave elementary reasons for consideration support region of polar quantization. In this paper the simple and complete asymptotical optimal analysis is given for constructing nonuniform unrestricted polar quantizer. We also gave the conditions for optimality of the nonuniform polar quantizer. We gave an equation for optimal number of points for different levels and also, optimal number of levels (these equations always repeating our optimization method for different  $r_{\text{max}}$ <br>
if the received the B. Sc. degree in electronics and<br>
and choosing the values for which  $D = D_x + D_o$  is the<br>
communications from the Faculty of Electronic solid<br>
min 1 L  $\sum_{i=1}^{n} P_{iopt} = N$ ). The equation for from the University of Ni  $D_{\sigma}^{opt}$  is given in a closed form. Applying our algorithm, incompleteness from [2] is eliminated.

#### **REFERENCES**

- Optimal, Fixed-Rate Scalar Quantizers" IEEE Transaction on Information Theory, vol.47, pp. 2972-2982, November 2001.
- 1.209 1.348 0.286 0.284 28.31 27 26.68 Performance of Unrestricted Polar Quantizer, IEEE Transactions on Information Theory, vol. 32, pp. 330-333, 1986.
	- [3] F. T. Arslan, "Adaptive Bit Rate Allocation in Compression of SAR Images with JPEG2000", The University of Arizona, USA, 2001.
	- Matlab Programming, John Wiley, New York., U.S.A, 2002.
	- [5] D. Hui, D. L. Neuhoff, "Asymptotic Analysis of Optimal Fixed-Rate Uniform Scalar Quantization," IEEE Transaction on Information Theory, vol.47, pp. 957-977, March 2001.
	- [6] Z. H. Peric, M. C. Stefanovic, "Asymptotic Analysis of Optimal Uniform Polar Quantization International Journal of Electronics and Communications, vol.56, pp. 345-347, 2002.
	- 34 28.31 27 26.68 Performance of Unrestricted Polar Quantizer",<br>  $\frac{128.31}{17} = \frac{300-4}{130} = \frac{300-321}{300.21}$  15. T. Arslam, "Adaptive Bit Rate Allocation in<br>  $\frac{1}{17} = \frac{300-4}{14} = \frac{1428}{14}$  14.28 [3] F. T. Arslam [7] Z. H. Peric, S. M. Bogosavljevic "An algorithm for construction of optimal polar quantizers", Journal of Electrical Engineering vol.4. No. 1 pp. 73-78, 2004.
		- L *IEEE Trans.Commun.*, vol. 40, pp. 1670-1674, [8] K. Popat and K. Zeger, "Robust quantization of memoryless sources using dispersive FIR filters," Nov. 1992.

## BIOGRAPHY

U.S.A, 2002.<br>
[5] D. Hui, D. L. Neuhoff, "Asymptotic Analysis of<br>
Optimal Fixed-Rate Uniform Scalar<br>
Quantization, *The F. Transaction on Information*<br>
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