# ASYMPTOTIC ANALYSIS OF OPTIMAL UNRESTRICTED POLAR QUANTIZATION

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**SUMMARY**

*The motivation for this work is maintaining high accuracy of phase information that is required for some applications such as interferometry and polarimetry, polar quantization techniques as well as their applications in areas such as computer holography, discrete Fourier transform encoding, and image processing. In this paper the simple and complete asymptotically analysis is given for a nonuniform polar quantizer with respect to the mean-square error (MSE) i.e. granular distortion ($D_g$). Granular (support) region of a quantizer is considered as the interval where quantization errors are small, or at least bounded; that's why it is greater challenge to include the overload distortion in estimation procedure of a quantizer ([1]). The support region for scalar quantizers has been found in [1] by minimization of the total distortion D, which is a combination of granular ($D_g$) and overload ($D_o$) distortions, $D=D_g+D_o$. Swaszek and Ku [2] didn't consider the problem of finding the optimal maximal amplitude, so-called, support region. The goal of this paper is solving the quantization problem in case of nonuniform polar quantizer and finding the corresponding support region. We also gave the conditions for optimum of the polar quantizer and optimal compressor function. The equation for $D_g^{opt}$ is given in a closed form. The construction procedure is given for i.i.d Gaussian source.*

***Keywords**: phase divisions, number of levels, optimal granular distortion, asymptotical analysis, Unrestricted Polar Quantization*

## 1. INTRODUCTION

Polar quantization techniques as well as their applications in areas such as computer holography, discrete Furrier transform encoding, image processing and communications have been studied extensively in the literature. Synthetic Aperture Radars (SARs) images can be represented in the polar format (i.e., magnitude and phase components) [3]. In the case of MSE quantization of a symmetric two-dimensional source, polar quantization gives the best result in the field of the implementation [3]. The motivation behind this work is to maintain high accuracy of phase information that is required for some applications such as interferometry and polarimetry, without loosing massive amounts of magnitude information [3].

One of the most important results in polar quantization are given by Swaszek and Ku who derived the asymptotically Unrestricted Polar Quantization (UPQ) [2]. Swaszek and Ku gave an asymptotic solution for this problem without a mathematical proof of the optimum and using, sometimes, quite hard approximations, which limit the application. Polar quantization consists of separate but uniform magnitude and phase quantization, on $N$ levels, so that rectangular coordinates of the source $(x,y)$ are transformed into the polar coordinates in the following form: $r=(x^2+y^2)^{1/2}$, where $r$ represents magnitude and $\phi$ is phase:

$$\phi=\begin{cases}\tan^{-1}\left(\frac{y}{x}\right)\\ \pi+\tan^{-1}\left(\frac{y}{x}\right)\\ \pi+\tan^{-1}\left(\frac{y}{x}\right)\\ 2\pi+\tan^{-1}\left(\frac{y}{x}\right)\end{cases}$$

for I, II, III and IV quadrant.

The asymptotic optimal quantization problem, even for the simplest case - uniform scalar quantization, is actually nowadays [5]. In [1] the analysis of scalar quantization is done in order to determine the optimal maximal amplitude.
Swaszek and Ku [2] didn't consider the problem of finding the optimal maximal amplitude, so-called, support region.

The support region for scalar quantizers has been found in [1] by minimization of the total distortion $D$, which is a combination of granular ($D_g$) and overload ($D_o$) distortions, $D=D_g+D_o$. The goal of this paper is solving the quantization problem in the case of nonuniform polar quantizer and finding the corresponding support region. It is done by analytical optimization of the granular distortion and numerical optimization of the total distortion.

In the paper Peric and Stefanovic [6] analyses are given for optimal asymptotic uniform polar quantization. Analysis of optimal polar quantization for moderate and smaller values of N is given in [7]. In this paper the simple and complete asymptotical analyses (for large values N) are given for a nonuniform polar quantizer with respect to the mean-square error (MSE) i.e. granular distortion ($D_g$). We consider $D$ as a function of the vector $\boldsymbol{P}=(P_i)_{1\le i\le L}$ whose elements are numbers of phase quantization levels at the each magnitude level. Said by different words, each concentric ring in quantization pattern is allowed to have a different number of partitions in the phase quantizer ($P_i$) when $r$ is in the $i$-th magnitude ring. Optimal Unrestricted Polar Quantization (OUPQ) must satisfy the constraint $\sum_{i=1}^{L} P_i = N$ in order to use all of $N$ regions for the quantization. We prove the existence of one minimum and derive the expression for evaluating $P_{opt}(\boldsymbol{r},\boldsymbol{m})$ for fixed values of reconstruction levels ($\boldsymbol{m}=(m_i)_{1\le i\le L}$), decision levels ($\boldsymbol{r}=(r_i)_{1\le i\le L+1}$) and number of levels $L$. We also gave the conditions for optimum of the polar quantizer, optimal compressor function and optimal numbers of levels. We derive $D_g^{opt}$ in a closed form.

We also gave the example of quantizer constructing for a Gaussian source. This case has the importance because of using Gaussian quantizer on an arbitrary source; we can take advantage of the central limit theorem and the known structure of an optimal scalar quantizer for a Gaussian random variable to encode a general process by first filtering it in order to produce an approximately Gaussian density, scalar-quantizing the result, and then inverse-filtering to recover the original [8].

## 2. CONDITIONS FOR OPTIMALITY AND DESIGN OF UNRESTRECTED POLAR QUANTIZER

For these analysis we assume that the input is from a continuously valued circularly source with unit variance rectangular coordinate marginals and bivariate density function $f(x,y)=p(\sqrt{x^2+y^2})$. Transforming to polar coordinates, the phase is uniformly distributed on a $[0,2\pi)$ and the magnitude is distributed on a $[0,\infty)$ with density function $f(r)=2\pi r p(r)$. Note that magnitude and phase are independent random variables. The transformed probability density function for the Gaussian source is $f(r,\phi)=\frac{1}{2\pi\sigma^2}\cdot re^{\frac{-r^2}{2\sigma^2}}=\frac{f(r)}{2\pi}$. Without loosing generality we assume that variance is: $2\sigma^2=1$.

We consider nonuniform polar quantizer with $L$ magnitude levels and $P_i$ phase reconstruction points at magnitude reconstruction level $m_i$, $1\le i\le L$. In order to minimize the distortion we proceed as follows.

First we partition the magnitude range $[0,r_{L+1}]$ into magnitude rings by $L+1$ decision levels (see **Fig. 1**) $\boldsymbol{r}=(r_1,...,r_{L+1})$ and $(0=r_1<r_2<...<r_L<r_{L+1}=r_{\max})$.

The magnitude reconstruction levels (see **Fig. 1**) $\boldsymbol{m}=(m_1,...,m_L)$ obviously satisfy $(0<m_1<m_2<...<m_L)$. Next we partition each magnitude ring into $P_i$ phase subdivisions. Let $\phi_{ij}$ and $\phi_{i,j+1}$ be two phase decision levels, and let $\psi_{i,j}$ be $j$-th phase reconstruction level for the $i$-th magnitude ring, $1\le j\le P_i$. Then $\phi_{i,j}=(j-1)2\pi/P_i \quad j=1,2,...,P_i+1$, and $\psi_{i,j}=(2j-1)\pi/P_i$ (see **Fig. 1**).

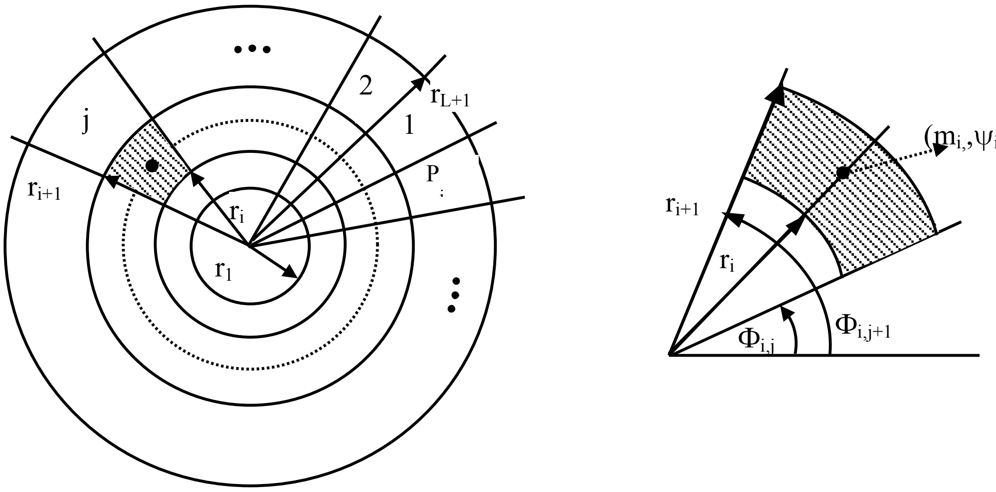


**Fig. 1** UPQ and j-th cell on i-th level preview

The distortion $D$ for UPQ ($r_{L+1}=\infty$) is [6]:

$$D=\frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{P_i}\int_{\phi_{i,j}}^{\phi_{i,j+1}}\int_{r_i}^{r_{i+1}}[r^2+m_i^2-2rm_i\cos(\phi-\psi_{i,j})]\frac{f(r)}{2\pi}drd\phi \quad (1)$$

Total distortion $D$, for OUPQ ($r_{L+1}=r_{\max}$) is a combination of granulation and overload distortions $D=D_g+D_o$:

$$D=\frac{1}{2}\sum_{i=1}^{L}\sum_{j=1}^{P_i}\int_{\phi_{i,j}}^{\phi_{i,j+1}}\int_{r_i}^{r_{i+1}}[r^2+m_i^2-2rm_i\cos(\phi-\psi_{i,j})]\frac{f(r)}{2\pi}drd\phi+\frac{1}{2}\sum_{j=1}^{P_L}\int_{\phi_{L,j}}^{\phi_{L,j+1}}\int_{r_{L+1}}^{\infty}[r^2+m_L^2-2rm_L\cos(\phi-\psi_{L,j})]\frac{f(r)}{2\pi}drd\phi \quad (2)$$

We integrated (2) by $\phi$, and get the equation for granular distortion:

$$D_g(P_1,\cdots,P_L)=\frac{1}{2}\sum_{i=1}^{L}\int_{r_i}^{r_{i+1}}[r^2+m_i^2-2rm_i\,\mathrm{sinc}(\frac{\pi}{P_i})]f(r)dr \quad (3)$$

(where in sinc($x$)=sin($x$)/$x$); (2) we use :

$$\frac{\sin(x)}{x}=1-\frac{1}{6}x^2+\varepsilon(x)$$

$$D_g\approx\frac{1}{2}\sum_{i=1}^{L}\int_{r_i}^{r_{i+1}}[(r-m_i)^2+\frac{rm_i}{3}\frac{\pi^2}{P_i^2}]f(r)dr \quad (4)$$

From: $\frac{\partial D_g}{\partial m_i}=0$

we can find $m_i$ as :

$$m_i=\left(1-\frac{1}{6}\frac{\pi^2}{P_i^2}\right)\frac{r_{i+1}+r_i}{2} \quad (5)$$

As final result, we find approximation for $m_i$ as:

$$m_i=\frac{r_{i+1}+r_i}{2} \quad (6)$$

We can obtain from High Resolution Theory [1] that high values for $R$ ($R=\log_2 N$) and critical values for $P_i$ satisfy given approximation.
The equation for $D_g$ is obtained by using High Resolution Theory [6].

$$D_g=\sum_{i=1}^{L}\frac{f(m_i)\Delta_i^3}{24}+\sum_{i=1}^{L}\frac{m_i^2\pi^2 f(m_i)\Delta_i}{6P_i^2} \quad (7)$$

where is $\Delta_i=r_{i+1}-r_i$.

We prove that the problem of minimizing the $D_g(\boldsymbol{P})$ is a convex programming problem. Function $D_g(\boldsymbol{P})$ is convex if its Hessian matrix is the positive semidefinite one [4].

$$\frac{\partial D_g}{\partial P_i}=-\frac{2\pi^2}{6(P_i)^3}m_i^2 f(m_i)\Delta_i$$

$$\frac{\partial^2 D_g}{\partial P_i\partial P_j}=\begin{cases}\frac{\pi^2}{(P_i)^4}m_i^2 f(m_i)\Delta_i\ ,\ i=j\\ 0,\ i\neq j\end{cases}$$

$$\Rightarrow\frac{\partial^2 D_g}{\partial P_i\partial P_j}\geq 0 \quad (8)$$

it follows that $D_g(\boldsymbol{P})$ is a convex function of $P$.

The minimization of function $D_g(\boldsymbol{P})$ for fixed number of magnitude levels $L$ constrained by the total number of reconstruction points $N$ is formulated in this way: minimize $D_g(\boldsymbol{P})$ under the constraints $\sum_{i=1}^{L}P_i=N$. We use the equation: $J=D_g+\lambda\sum P_i$, where $\lambda$ represents Lagrange multiplier. From $\frac{\partial J}{\partial P_i}=0$ we obtain :

$$\frac{\partial J}{\partial P_i}=-\frac{2\pi^2}{6(P_i)^3}m_i^2 f(m_i)\Delta_i+\lambda\ ,$$

and finally:

$$P_{iopt}=N\frac{\sqrt[3]{m_i^2 f(m_i)\Delta_i}}{\sum_{j=1}^{L}\sqrt[3]{m_j^2 f(m_j)\Delta_j}};\qquad 1\leq i\leq L\ . \quad (9)$$

The formula (9) is like to formula in paper [7] (i.e. it should obtained utilizing approximation

$\int_{r_i}^{r_{i+1}}rf(r)dr\approx m_i f(m_i)\Delta_i$ ).

The approximation given by Swaszek and Ku for the asymptotically Unrestricted Polar Quantization (UPQ) [2]:

$$r_{L+1}-m_L\approx m_L-r_L=\frac{1}{2Lg'(m_L)} \quad (10)$$

is not correct for Unrestricted Polar Quantization because $r_{L+1}-m_L\to\infty$. That is the elementary reason for introducing support region ($r_{\max}$), where $r_{\max}$ is restricted for the scalar quantization analysis, which is based on using compressor function $g$.

We replaced $\Delta_i=\frac{r_{\max}}{Lg'(m_i)}$, where $g$ is compressor function, and approximate the sums by integrals ($\Delta_i\approx dr$), and we get $P_i$ as:

$$P_i\approx\frac{Nr_{\max}\sqrt[3]{m_i^2 f(m_i)/g'(m_i)}}{L\int_0^{r_{\max}}\sqrt[3]{r^2 f(r)(g'(r))^2}dr}\ . \quad (11)$$

As final result, we find the equation for granular distortion:

$$D_g = \frac{r_{max}^2}{24L^2}\int_0^{r_{max}} \frac{f(r)}{(g'(r))^2}dr + \frac{\pi^2 L^2}{6N^2 r_{max}^2}\left(\int_0^{r_{max}} \sqrt[3]{r^2 f(r)(g'(r))^2}\,dr\right)^3 = \frac{r_{max}^2}{24L^2}I_0 + \frac{\pi^2 L^2}{6N^2 r_{max}^2}I^3 \qquad (12)$$

The function $D_g(L)$ is convex of $L$, because $\frac{\partial^2 D_g}{\partial L^2} = \frac{r_{max}^2}{4L^4}I_0 + \frac{\pi^2}{3N^2 r_{max}^2}I^3$. The optimal number of levels problem can be solved analytically only for the asymptotical analysis as it is suggested: from the condition $\frac{\partial D_g}{\partial L} = 0$ we came to the optimal solution for $L_{opt}$:

$$L_{opt} = r_{max}\sqrt[4]{\frac{I_0 N^2}{4\pi^2 I^3}} \qquad (13)$$

The optimal granular distortion is:

$$D_g^{opt} = \frac{\pi}{6N} I\sqrt{I_0 I} \qquad (14)$$

We can obtain $g(r)$ like in [2] by using Hölder's inequality:

$$g(r) = \left(r_{max}\int_0^r \sqrt[4]{\frac{f(r)}{r}}dr\right) \Big/ \left(\int_0^{r_{max}} \sqrt[4]{\frac{f(r)}{r}}dr\right) \qquad (15)$$

and $$D_g^{opt} = \frac{\pi}{6N}\left(\int_0^{r_{max}} \sqrt{rf(r)}\,dr\right)^2. \qquad (16)$$

*Example:*

We compared results for Gaussian source. Numbers of magnitude levels and reconstruction points, reconstruction points and decision levels are calculated by using (for Gaussian source [2]):

$$L = \sqrt{N/2}$$

$$P_i = \sqrt{\pi}N^{1/2} m_i \exp\left(-\frac{m_i^2}{8}\right)$$

$$r_i = g^{-1}[(i-1)/L],\ 1 \le i \le L; \quad r_{L+1} = \infty$$

$$m_i = g^{-1}[(2i-1)/2L],\ 1 \le i \le L$$

$g(r)$ is a compressor function given by :

$$g(r) = \left(\int_0^r \sqrt[4]{\frac{f(s)}{s}}ds\right) \Big/ \left(\int_0^{\infty} \sqrt[4]{\frac{f(s)}{s}}ds\right)$$

Method presented in the paper [2] cann't be applied for some values of $N$ and numbers of level $L$. For number of level $L$, the total number of points is in the range,

$(\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor),\ N_1 = 2(round(L)-0.5)^2,\ N_2 = 2(round(L)+0.5)^2$.

This follows from the fact that ***r*** and ***m*** are equal for any $N$ in the range($\lceil N_1 \rceil \le N \le \lfloor N_2 \rfloor$), and since $P_{opt}$ is dependent of ***m***, $N$ and introduced approximations, then $\sum_{i=1}^{L} P_i = N$ will not be satisfied. In addition, for some values of $N$ from the former range, we cannot reach $\sum_{i=1}^{L} P_i = N$.

With goal to calculate rough (approximately) the deviation of calculated number of points than proposed number of points N by the method from paper [2], we will make next approximate analisys. For estimation of $\sum_{i=1}^{L} P_i$ we gave following approximation: we found the total number of points [2] as:

$$\sum_{i=1}^{L} P_i = \sum_{i=1}^{L} \sqrt{\pi N} m_i \exp\left(-\frac{m_i^2}{8}\right)\frac{\Delta_i}{\Delta_i} \approx$$

$$\approx round(L)\sum_{i=1}^{L}\sqrt{\frac{N}{2}} m_i \exp\left(-\frac{m_i^2}{4}\right)\Delta_i \approx$$

$$\approx round(L)\sqrt{\frac{N}{2}}\int_0^{\infty} r\exp\left(-\frac{r^2}{4}\right)dr =$$

$$= round(L)\sqrt{2N} = M$$

We considered the most critical values for $N = M_1 = \lceil N_1 \rceil$ and $N = M_2 = \lfloor N_2 \rfloor$ where $\delta_i = |M_i - M|$. (see Table 1.)

| $L$ | $M_1$ | $M_2$ | $\delta_1 = \lvert M_1 - M\rvert$ | $\delta_2 = \lvert M_2 - M\rvert$ |
|---|---|---|---|---|
| 11 | 221 | 264 | 10.26 | 11.24 |
| 50 | 4901 | 5100 | 44.25 | 50.25 |
| 100 | 19801 | 20200 | 99.25 | 100.25 |
| 150 | 44701 | 45300 | 149.25 | 150.25 |
| 200 | 79601 | 80400 | 199.25 | 200.25 |

Table 1.

Correct analysis i.e the deviation of calculated number of points than proposed number of points we will give for L=11 i N=221. (see Table 2.)

By Swaszek and Ku [2] for each L=const, **m** and **r** are equal. For $N = \lceil N_1 \rceil = 221 \Rightarrow$ L=11 $\sum_{i=1}^{L} P_i = 232{,}84$, and $\delta_1 = 11{,}84$ (approximately $\delta_1 = 10{,}26$ from Table 1).

For $P_i = round(P_i)$ we can't satisfy constraint $\sum_{i=1}^{L} P_i = 233 \ne N = 221$. We get 11 values for $P_i$ by rounding, but 9 of them are different from values in [2].

| $r_i[2]$ | $m_i[2]$ | $\Delta i[2]$ | $\Delta iopt$ | $P_i[2]$ | $P_{opt}$ | $P_{real}$ |
|---|---|---|---|---|---|---|
| 0 | 0.114 | 0.228 | 0.227 | 2.991 | 3 | 2.822 |
| 0.228 | 0.343 | 0.231 | 0.230 | 8.901 | 8 | 8.392 |
| 0.459 | 0.577 | 0.238 | 0.236 | 14.58 | 14 | 13.74 |
| 0.697 | 0.819 | 0.248 | 0.247 | 19.85 | 19 | 18.70 |
| 0.945 | 1.074 | 0.264 | 0.262 | 24.50 | 23 | 23.09 |
| 1.209 | 1.348 | 0.286 | 0.284 | 28.31 | 27 | 26.68 |
| 1.495 | 1.651 | 0.322 | 0.318 | 30.94 | 29 | 29.19 |
| 1.817 | 1.996 | 0.377 | 0.371 | 31.96 | 30 | 30.21 |
| 2.194 | 2.415 | 0.477 | 0.465 | 30.69 | 29 | 29.12 |
| 2.671 | 2.980 | 0.711 | 0.673 | 25.88 | 25 | 24.77 |
| 3.382 | 4.002 | ∞ | 1.551 | 14.24 | 14 | 14.28 |

Table 2.

For a fixed number N we determine ( $P_i$ , $L$ )

Step 1)

$$L_{opt} = r_{\max} \sqrt[4]{\frac{I_0 N^2}{4\pi^2 I^3}} ;$$

$g(r)$ is a compressor function given by :

$$g(r) = (r_{\max} \int_0^r \sqrt[4]{\frac{f(r)}{r}} dr) \Big/ ( \int_0^{r_{\max}} \sqrt[4]{\frac{f(r)}{r}} dr)$$

Step 2)

$$P_{iopt} = N \frac{\sqrt[3]{m_i^2 f(m_i) \Delta_i}}{\sum_{j=1}^{L} \sqrt[3]{m_j^2 f(m_j) \Delta_j}}; \qquad 1 \le i \le L$$

Step 3)

The exact optimal value for $r_{\max}$ is obtained repeating our optimization method for different $r_{\max}$ and choosing the values for which $D = D_g + D_o$ is minimal.

## 3. CONCLUSION

The solution given by Swaszek and Ku[2] is the best one found by now but for large N. Swaszek and Ku gave an asymptotic solution for unrestricted nonuniform polar quantization without a mathematical proof of the optimum and using, sometimes, quite hard approximations, which limit the application. We gave elementary reasons for consideration support region of polar quantization. In this paper the simple and complete asymptotical optimal analysis is given for constructing nonuniform unrestricted polar quantizer. We also gave the conditions for optimality of the nonuniform polar quantizer. We gave an equation for optimal number of points for different levels and also, optimal number of levels (these equations always satisfy the constraint: $\sum_{i=1}^{L} P_{iopt} = N$ ). The equation for $D_g^{opt}$ is given in a closed form. Applying our algorithm, incompleteness from [2] is eliminated.

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